Supplementary Material to "Dependent Microstructure Noise and Integrated Volatility Estimation from High-Frequency Data"

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Abstract

Section A of this appendix contains detailed proofs of our results. In Sections B and C, we provide additional Monte Carlo simulation studies and empirical results.

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A Proofs

A.1 Assumptions and Notation

In all proofs that follow, the constants C may vary from line to line, or even within one line. We add a subscript *par* if they depend on some parameter *par*. In the sequel, we will employ Lemma VIII.3.102 in Jacod and Shiryaev (2003) repeatedly, and we will refer to it as the *JS-Lemma*.

Adopting the standard localization procedure (see e.g., Jacod and Protter (2011) for further details), we may assume that:

Assumption A.1. *The efficient price X satisfies the Assumption 2.1 with b_t and σ_t bounded (uniformly in ω and t).*

This implies that for all stopping times $0 \leq S \leq T \leq 1$ we have

$$\begin{aligned} &\mathbb{E}\left(\left|X_T - X_S\right|^p \left|\mathcal{F}_S\right.\right) \leq C_p \mathbb{E}\left(T - S \left|\mathcal{F}_S\right.\right), \quad \forall p \geq 2. \qquad \text{(A.1)} \\ &\left|\mathbb{E}\left(X_T - X_S \left|\mathcal{F}_S\right.\right)\right| \leq C \mathbb{E}\left(T - S \left|\mathcal{F}_S\right.\right). \end{aligned}$$

We first introduce some notation that is used to prove the results in Section 4.1:

$$\begin{aligned} G_i^n(s) &:= \sum_{j=1}^{k_n - 1} g_j^n \mathbf{1}_{\{((i+j-1)\Delta_n, (i+j)\Delta_n]\}}(s); \\ \mathcal{H}_i^n &:= \mathcal{F}_i^n \otimes \mathcal{G}_i; \\ \beta_m^n &:= n^{1/4} \left(\sigma_{mk_n\Delta_n} \overline{W}_{mk_n}^n + \overline{U}_{mk_n}^n \right); \\ \xi_m^n &:= n^{1/4} \overline{Y}_{mk_n}^n - \beta_m^n; \\ \eta_m^n &:= \frac{n^{r/4}}{\theta} \mathbb{E}\left(\left| \overline{Y}_m^n \right|^r \left| \mathcal{H}_{mk_n}^n \right. \right); \\ \widetilde{\eta_m^n} &:= \frac{\mu_r}{\theta} \left(\theta \psi_0 \sigma_{mk_n\Delta_n}^2 + \frac{\psi_1}{\theta} \Sigma_U \right)^{\frac{r}{2}}; \\ \mathrm{PAV}^n &:= \sum_{m=0}^{M_n - 1} \eta_m^n; \\ \widetilde{\mathrm{PAV}}^n &:= \sum_{m=0}^{M_n - 1} \widetilde{\eta_m^n}. \end{aligned}$$

To prove the results presented in Section 4.2 we will also need the following:

$$\begin{aligned} \widehat{G}_i^n(j, j') &= \int_0^\infty G_{i+j}^n(s) G_{i+j'}^n(s) \mathrm{d}s, \\ \overline{G}_i^n(j, j') &= \int_0^\infty G_{i+j}^n(s) G_{i+j'}^n(s) \mathrm{d}s \int_0^s G_{i+j}^n(u) G_{i+j'}^n(u) \mathrm{d}u, \\ X_i^n(t) &= B_i^n(t) + M_i^n(t); \end{aligned}$$

where $B_i^n(t) = \int_0^t b_s G_i^n(s)\mathrm{d}s$, and $M_i^n(t) = \int_0^t \sigma_s G_i^n(s)\mathrm{d}W_s$. Furthermore, we define

$$\begin{aligned}
&\mathcal{K}_i^n = \mathcal{F}_i^n \otimes \mathcal{G}_{i-\lfloor \frac{k_n}{2} \rfloor}, \quad \mathcal{J}(p)_j^n = \mathcal{K}_{j(p+1)k_n}^n, \quad \mathcal{J}'(p)_j^n = \mathcal{K}_{j(p+1)k_n+pk_n}^n, \quad \mathrm{IV}_t = \int_0^t \sigma_s^2 \mathrm{d}s; \\
&\widehat{c}_i^n = \sum_{j=1}^{k_n-1} \left(g_j^n\right)^2 \Delta_{i+j}^n \mathrm{IV}, \quad \alpha_n = \mathbb{E}\Big(\left(\overline{U}_i^n\right)^2\Big), \quad \widehat{U}_i^n = \left(\overline{U}_i^n\right)^2 - \alpha_n, \quad \widehat{X}_i^n = (\overline{X}_i^n)^2 - \widehat{c}_i^n; \\
&\Psi_i^n = \left(\overline{Y}_i^n\right)^2 - \widehat{c}_i^n - \alpha_n = \widehat{X}_i^n + \widehat{U}_i^n + 2\overline{X}_i^n \overline{U}_i^n, \quad \zeta(p)_i^n = \sum_{j=i}^{i+pk_n-1} \Psi_j^n; \\
&\eta(p)_j^n = \frac{\sqrt{\Delta_n}}{\theta\psi_0} \zeta(p)_{j(p+1)k_n}^n, \quad \overline{\eta}(p)_j^n = \mathbb{E}\left(\eta(p)_j^n \,\middle|\, \mathcal{J}(p)_j^n\right); \\
&\eta'(p)_j^n = \frac{\sqrt{\Delta_n}}{\theta\psi_0} \zeta(1)_{j(p+1)k_n+pk_n}^n, \quad \overline{\eta}'(p)_j^n = \mathbb{E}\left(\eta'(p)_j^n \,\middle|\, \mathcal{J}'(p)_j^n\right),
\end{aligned}$$

and let $K_n^p = \lfloor \frac{1}{(p+1)k_n\Delta_n} \rfloor - 1, I_n^p = (K_n^p + 1)(p+1)k_n$. We can then decompose $\widetilde{\mathrm{IV}}_n - \mathrm{IV}$ into the following terms:

$$\begin{aligned}
&F(p)_n = \sum_{j=0}^{K_n^p} \overline{\eta}(p)_j^n, \quad M(p)_n = \sum_{j=0}^{K_n^p} \left(\eta(p)_j^n - \overline{\eta}(p)_j^n\right); \\
&F'(p)_n = \sum_{j=0}^{K_n^p} \overline{\eta}'(p)_j^n, \quad M'(p)_n = \sum_{j=0}^{K_n^p} \left(\eta'(p)_j^n - \overline{\eta}'(p)_j^n\right); \\
&\widehat{C}(p)_n = \frac{\sqrt{\Delta_n}}{\theta\psi_0} \sum_{i=I_n^p}^{n-k_n+1} \Psi_i^n; \\
&\widehat{C}'(p)_n = \frac{(n-k_n+2)\,\alpha_n\sqrt{\Delta_n}}{\theta\psi_0} - \frac{\psi_1}{\theta^2\psi_0}\left(\widehat{\gamma(0)}_n + 2\sum_{j=1}^{\ell_n} \widehat{\gamma(j)}_n\right); \\
&\widehat{C}''_n = \frac{\sqrt{\Delta_n}}{\theta\psi_0} \sum_{i=0}^{n-k_n+1} \widehat{c}_i^n - \mathrm{IV},
\end{aligned}$$

since we have

$$\widetilde{\mathrm{IV}}_n - \mathrm{IV} = M(p)_n + M'(p)_n + F(p)_n + F'(p)_n + \widehat{C}(p)_n + \widehat{C}'(p)_n + \widehat{C}''_n. \tag{A.2}$$

A.2 Auxiliary Lemmas

We will often need the following two useful results based on the JS-Lemma.

Let Z be an integrable random variable with finite variance and measurable with respect to $\mathcal{G}_{k'+k}$ (see Assumption 2.2 for the definition of this σ-algebra) and define

$$C_Z^k := \mathbb{E}\Big(\left(\mathbb{E}\left(Z \,|\mathcal{G}_{k'}\right) - \mathbb{E}(Z)\right)^2\Big), \quad \Lambda_Z := \frac{\mathbb{E}\left(Z \,|\mathcal{G}_{k'}\right) - \mathbb{E}(Z)}{\sqrt{C_Z^k}}.$$

Then we have by the JS-Lemma

$$\mathbb{E}\left(Z\,|\mathcal{G}_{k'}\right) = \mathbb{E}(Z) + \Lambda_Z\sqrt{C_Z^k}, \tag{A.3}$$

with $\mathbb{E}\big(\Lambda_Z^2\big) = 1$ and $C_Z^k \le Ck^{-2v}$.

Another application of the JS-Lemma gives that if Z_i, Z_j are $\mathcal{G}_i$- and $\mathcal{G}_j$-measurable random variables respectively, with mean zero and bounded variance, then we have for all $k \le i < j$ that

$$\mathbb{E}(|\mathbb{E}\left(Z_i Z_j\,|\mathcal{G}_k\right)|) \le C\,(j-i)^{-v}\,. \tag{A.4}$$

To see this, we use the JS-Lemma to obtain (since the Z_j have bounded variance):

$$c_{ij} := \mathbb{E}\Big(\left(\mathbb{E}\left(Z_j\,|\mathcal{G}_i\right)\right)^2\Big) \le C\,(j-i)^{-2v}\,. \tag{A.5}$$

Then,

$$\mathbb{E}(|\mathbb{E}\left(Z_i Z_j\,|\mathcal{G}_k\right)|) \le \sqrt{C\,(j-i)^{-2v}}\,\mathbb{E}\left(\left|\mathbb{E}\left(Z_i\frac{\mathbb{E}\left(Z_j\,|\mathcal{G}_i\right)}{\sqrt{c_{ij}}}\,|\mathcal{G}_k\right)\right|\right).$$

Now applying the Cauchy-Schwarz inequality and using the fact that the variance of the Z_i is bounded, we obtain (A.4).

Next, in the setting of Section 4, we recall some useful estimates (see Jacod et al. (2009)) for pre-averaged sequences defined in (19):

$$\left|\mathbb{E}\left(\overline{X}_i^n\,|\mathcal{F}_i^n\right)\right| \le C\sqrt{\Delta_n}, \quad \mathbb{E}\left(\left|\overline{X}_i^n\right|^q|\mathcal{F}_i^n\right) \le C_q\Delta_n^{q/4}, \tag{A.6}$$

for any $q > 0$, and

$$\mathbb{E}\left(\left(\overline{W}_i^n\right)^2|\mathcal{F}_i^n\right) = k_n\Delta_n\psi_0 + O_p(\Delta_n^{3/4}). \tag{A.7}$$

The following lemma, which establishes a central limit theorem for general pre-averaged noise, plays a central role in the proofs of the results in Sections 4 and 5.

Lemma A.1. *Assume that the noise satisfies Assumption 2.2 and that (14) is satisfied. Then, the following central limit theorem holds for $\overline{U}_i^n$:*

$$n^{1/4}\overline{U}_i^n \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\psi_1\Sigma_U}{\theta}\right). \tag{A.8}$$

Proof. Let $a_j^n = -\overline{g}_j^n\sqrt{k_n/\phi_1^n(0)}$. First, a Riemann sum approximation implies

$$\phi_1^n(0) = \psi_1 + o(\Delta_n^{1/4}). \tag{A.9}$$

Next, for any $\ell \in \mathbb{Z}$, the Lipschitz property of g' implies $\left|\overline{g}_j^n - \overline{g}_{j-\ell}^n\right| \leq C|\ell|k_n^{-2}$, so

$$|\phi_1^n(\ell) - \psi_1| \leq C|\ell|/k_n + o(\Delta_n^{1/4}). \tag{A.10}$$

Since $\mathbb{E}\left(\left(\sum_{j=0}^{k_n-1} a_j^n U_{i+j}\right)^2\right) = \frac{1}{\phi_1^n(0)} \sum_{|\ell|\leq k_n} \phi_1^n(\ell)\gamma(\ell)$, we have

$$\left|\mathbb{E}\left(\left(\sum_{j=0}^{k_n-1} a_j^n U_{i+j}\right)^2\right) - \frac{1}{\phi_1^n(0)} \sum_{|\ell|\leq k_n} \psi_1\gamma(\ell)\right| \leq \frac{C}{\phi_1^n(0)} \sum_{|\ell|\leq k_n} \frac{|\gamma(\ell)\ell|}{k_n} \leq C\sqrt{\Delta_n}, \tag{A.11}$$

where we used that $|\gamma(\ell)\ell| \leq C|\ell|^{1-v}$ with $v > 2$, and $k_n = O(n^{1/2})$. Then $\sum_{|\ell|>k_n} \gamma(l) \leq C k_n^{1-v}$ gives

$$\left|\frac{1}{\phi_1^n(0)} \sum_{|\ell|\leq k_n} \psi_1\gamma(\ell) - \Sigma_U\right| \leq C\sqrt{\Delta_n}, \tag{A.12}$$

and we see that $\mathbb{E}\left(\left(\sum_{j=0}^{k_n-1} a_j^n U_{i+j}\right)^2\right) \to \Sigma_U$.

Since we assume the existence of moments of noise of all orders, and $v > 1$, we have for sufficiently large r that $v - \frac{2}{r-2} > 1$, which implies

$$\sum_{k\in\mathbb{N}^*} k^{\frac{2}{r-2}} \rho_k < \infty,$$

where the $\{\rho_k\}$ are the ρ-mixing coefficients. This is sufficient for the following CLT, according to Rio (1997)[1]:

$$\sum_{j=0}^{k_n-1} a_j^n U_{i+j} \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \Sigma_U\right).$$

Since $n^{1/4}\overline{U}_i^n = \sqrt{\frac{\phi_1^n(0)}{\Delta_n^{1/2} k_n}} \sum_{j=0}^{k_n-1} a_j^n U_{i+j}$, we obtain by (A.11) and (A.12), using (18) and (A.9), that

$$\mathbb{E}\left(\left(n^{1/4}\overline{U}_i^n\right)^2\right) = \frac{\psi_1\Sigma_U}{\theta} + o(\Delta_n^{1/4}), \tag{A.13}$$

and the stated result follows. □

The result of Lemma A.1 for the asymptotics of pre-averaged noise will allow us to prove the results in Subsection 4.1 using a similar strategy as in Podolskij and Vetter (2009a,b). However, their proofs need to be modified for our setting. The following lemmas will therefore turn out to be useful.

Lemma A.2. *Assume the conditions of Theorem 4.1 are satisfied. Then there is, for any $q > 0$, some*

[1] Rio (1997) discusses *strongly mixing* or α-*mixing*, which is implied by ρ-mixing.

constant $C_q > 0$ (depending on q), such that for all m:

$$\mathbb{E}(|\xi_m^n|^q) + \mathbb{E}\Big(\Big|n^{1/4}\overline{X}_i^n\Big|^q\Big) < C_q; \tag{A.14}$$

$$\mathbb{E}(|\beta_m^n|^q) + \mathbb{E}\Big(\Big|n^{1/4}\overline{Y}_i^n\Big|^q\Big) < C_q. \tag{A.15}$$

Proof of Lemma A.2. The boundedness of the moments of $n^{1/4}\overline{X}_i^n$ follows from (A.6), which also establishes the same bound for $n^{1/4}\overline{W}_i^n$ if we take the drift of X equal to zero and the volatility constant. This, together with the boundedness of σ, gives the boundedness of $\mathbb{E}(\xi_m^n)$ since we can write $\xi_m^n = n^{1/4}\left(\overline{X}_{mk_n}^n - \sigma_{mk_n\Delta_n}\overline{W}_{mk_n}^n\right)$.

Now we show the boundedness of $\mathbb{E}\Big(\Big|n^{1/4}\overline{Y}_i^n\Big|^q\Big)$. Hölder's inequality implies

$$\mathbb{E}\Big(\Big|n^{1/4}\overline{Y}_i^n\Big|^q\Big) \le C_q\left(\mathbb{E}\Big(\Big|n^{1/4}\overline{X}_i^n\Big|^q\Big) + \mathbb{E}\Big(\Big|n^{1/4}\overline{U}_i^n\Big|^q\Big)\right).$$

Boundedness of $\mathbb{E}\Big(\Big|n^{1/4}\overline{X}_i^n\Big|^q\Big)$ has already been established, while $\mathbb{E}\Big(\Big|n^{1/4}\overline{U}_i^n\Big|^q\Big)$ is known to be bounded by Lemma A.1 and the well known fact that convergence in distribution implies convergence in moments under a uniformly bounded moments condition, see, e.g., Theorem 4.5.2 of Chung (2001). The result for $\mathbb{E}(|\beta_m^n|^q)$ follows by similar arguments. □

Lemma A.3. *Assume the conditions of Theorem 4.1 are satisfied. Then we have for all even integers $r > 2$ that, uniformly in m,*

$$\mathbb{E}\left((\beta_m^n)^2\,\middle|\mathcal{H}_{mk_n}^n\right) = \left(\theta\psi_0\sigma_{mk_n\Delta_n}^2 + \frac{\psi_1\Sigma_U}{\theta}\right) + o_p(n^{-1/4}), \tag{A.16}$$

$$\mathbb{E}\left((\beta_m^n)^r\,\middle|\mathcal{H}_{mk_n}^n\right) = \mu_r\left(\theta\psi_0\sigma_{mk_n\Delta_n}^2 + \frac{\psi_1\Sigma_U}{\theta}\right)^{r/2} + o_p(1), \tag{A.17}$$

with μ_r the moment of order r of a standard normal random variable.

Proof. Let $\{r_n\}$ be a sequence of integers satisfying

$$r_n \asymp n^{\vartheta}, \quad \frac{1}{4v} < \vartheta < \frac{1}{4}. \tag{A.18}$$

For any process Z, denote

$$\overline{Z}_{m,r_n}^n := -\sum_{j=0}^{r_n-1}\overline{g}_j^n Z_{mk_n+j}^n,$$

$$\overline{Z}_{r_n,m+1}^n := -\sum_{j=r_n}^{k_n-1}\overline{g}_j^n Z_{mk_n+j}^n.$$

Let

$$\overline{\beta}_{m,r_n}^n := n^{1/4}\overline{U}_{m,r_n}^n \qquad \overline{\beta}_{r_n,m+1}^n := n^{1/4}\left(\sigma_{mk_n\Delta_n}\overline{W}_{mk_n}^n + \overline{U}_{r_n,m+1}^n\right). \tag{A.19}$$

This implies that $\beta_m^n = \overline{\beta}_{m,r_n}^n + \overline{\beta}_{r_n,m+1}^n$. We first prove (A.16) by establishing the following three results:

$$\mathbb{E}\left((\beta_m^n)^2 \,|\mathcal{H}_{mk_n}^n\right) - \mathbb{E}\left(\left(\overline{\beta}_{r_n,m+1}^n\right)^2 |\mathcal{H}_{mk_n}^n\right) = o_p(n^{-1/4}), \tag{A.20}$$

$$\mathbb{E}\left(\left(\overline{\beta}_{r_n,m+1}^n\right)^2 |\mathcal{H}_{mk_n}^n\right) - \mathbb{E}\left(\left(\overline{\beta}_{r_n,m+1}^n\right)^2 |\mathcal{F}_{mk_n}^n\right) = o_p(n^{-1/4}), \tag{A.21}$$

$$\mathbb{E}\left(\left(\overline{\beta}_{r_n,m+1}\right)^2 |\mathcal{F}_{mk_n}^n\right) - \left(\theta\psi_0\sigma_{mk_n\Delta_n}^2 + \frac{\psi_1\Sigma_U}{\theta}\right) = o_p(n^{-1/4}). \tag{A.22}$$

1. To prove (A.20), it is enough to show that

$$\mathbb{E}\left(\left(\overline{\beta}_{m,r_n}^n\right)^2 |\mathcal{H}_{mk_n}^n\right) = o_p(n^{-1/4}), \tag{A.23}$$

$$\mathbb{E}\left(\left(\overline{\beta}_{r_n,m+1}^n\right)\left(\overline{\beta}_{m,r_n}^n\right) |\mathcal{H}_{mk_n}^n\right) = o_p(n^{-1/4}). \tag{A.24}$$

To establish (A.23), we write

$$\left(\overline{\beta}_{m,r_n}^n\right)^2 = n^{1/2}\left(\overline{U}_{m,r_n}^n\right)^2 = n^{1/2}\sum_{j=0}^{r_n-1}\sum_{j'=0}^{r_n-1} \overline{g}_j^n \overline{g}_{j'}^n U_{mk_n+j}^n U_{mk_n+j'}^n.$$

Taking conditional expectations we see that the left-hand side in (A.23) is smaller than

$$\sum_{j=0}^{r_n-1}(\overline{g}_j^n)^2\mathbb{E}\left((U_{mk_n+j})^2 \,|\mathcal{H}_{mk_n}^n\right) + 2\sum_{j=0}^{r_n-2}\sum_{j'=j+1}^{r_n-1}\left|\overline{g}_j^n\overline{g}_{j'}^n\right| \left|\mathbb{E}\left(U_{mk_n+j}U_{mk_n+j'} \,|\mathcal{H}_{mk_n}^n\right)\right|.$$

Since $\left|\overline{g}_j^n\right| \le C\sqrt{\Delta_n}$ for all j, we find by (A.4),

$$\mathbb{E}\left(\left(\overline{\beta}_{m,r_n}^n\right)^2 |\mathcal{H}_{mk_n}^n\right) \le C\sqrt{\Delta_n}\left(r_n + 2\sum_{j=0}^{r_n-2}\sum_{j'=j+1}^{r_n-1}(j'-j)^{-v}\right) \le C\sqrt{\Delta_n}r_n, \tag{A.25}$$

and this proves (A.23) due to (A.18). To prove (A.24) it is enough to show that

$$\sqrt{n}\mathbb{E}\left(\overline{U}_{m,r_n}^n \overline{U}_{r_n,m+1}^n \,|\mathcal{G}_{mk_n}\right) = O_p(n^{-1/2}), \tag{A.26}$$

$$n^{\frac{1}{4}}\mathbb{E}\Big(\mathbb{E}\left(\overline{\beta}_{m,r_n}^n \overline{W}_{mk_n}^n \,|\mathcal{H}_{mk_n}^n\right)\Big) = o_p(n^{-1/4}). \tag{A.27}$$

The first result follows since the left-hand side equals

$$\mathbb{E}\left(n^{1/2}\sum_{j=0}^{r_n-1}\sum_{j'=r_n}^{k_n-1}\overline{g}_j^n\overline{g}_{j'}^n U_{mk_n+j}^n U_{mk_n+j'}^n \mid \mathcal{G}_{mk_n}\right) \le Cn^{1/2}\sum_{j=0}^{r_n-1}\sum_{j'=r_n}^{k_n-1}\sqrt{\Delta_n}\sqrt{\Delta_n}\,|j'-j|^{-v}$$

by (A.4), and since $v > 2$ we get (A.26).

For (A.27), we note that the independence of $\mathcal{G}, \mathcal{F}$, and the estimates (A.6) and (A.23) imply

$$n^{\frac{1}{4}}\mathbb{E}\Big(\Big|\mathbb{E}\left(\overline{\beta}_{m,r_n}^n \overline{W}_{mk_n}^n \,|\mathcal{H}_{mk_n}^n\right)\Big|\Big) \le Cn^{\frac{1}{4}}\sqrt{\Delta_n}\mathbb{E}\Big(\Big|\mathbb{E}\left(\overline{\beta}_{m,r_n}^n \,|\mathcal{G}_{mk_n}\right)\Big|\Big) \le C\Delta_n^{\frac{1}{4}}\sqrt{n^{-\frac{1}{4}}}. \tag{A.28}$$

This proves (A.24) and hence (A.20) has now been established.

2. To prove (A.21) we note that the left-hand side of (A.21) is

$$\mathbb{E}\left(\left(n^{\frac{1}{4}}\overline{U}^n_{r_n,m+1}\right)^2 \big|\mathcal{H}^n_{mk_n}\right) - \mathbb{E}\left(\left(n^{\frac{1}{4}}\overline{U}^n_{r_n,m+1}\right)^2\right),$$

which is of order $O_p(r_n^{-v})$ by (A.3), so (A.21) follows from (A.18).

3. Finally, we prove (A.22). We have by Lemma 4 of Podolskij and Vetter (2009a) that

$$\mathbb{E}\left(\left(n^{1/4}\sigma_{mk_n\Delta_n}\overline{W}^n_{mk_n}\right)^2 \big|\mathcal{F}^n_{mk_n}\right) = n^{1/2}\sigma^2_{mk_n\Delta_n}k_n\Delta_n\psi_0 + o_p(n^{-1/4}) = \sigma^2_{mk_n\Delta_n}\psi_0\theta + o_p(n^{-1/4}),$$

where the last equality follows from (18). Due to the independence of $\mathcal{G}$ and $\mathcal{F}$ we therefore only need to show that

$$\mathbb{E}\left(\left(n^{1/4}\overline{U}^n_{r_n,m+1}\right)^2\right) = \frac{\psi_1\Sigma_U}{\theta} + o_p(n^{-1/4}).$$

We know from (A.13) that

$$\mathbb{E}\left(\left(n^{1/4}\overline{U}^n_{mk_n}\right)^2\right) = \frac{\psi_1\Sigma_U}{\theta} + o(\Delta_n^{1/4}),$$

so the desired result follows if we can show that

$$\left|\mathbb{E}\left(\left(n^{1/4}\overline{U}^n_{mk_n}\right)^2\right) - \mathbb{E}\left(\left(n^{1/4}\overline{U}^n_{r_n,m+1}\right)^2\right)\right| = o_p(n^{-1/4}). \tag{A.29}$$

But this follows from

$$\mathbb{E}\left(\left(n^{1/4}\overline{U}^n_{m,r_n}\right)^2\right) \le C\Delta_n r_n; \quad \mathbb{E}\left(\overline{U}^n_{m,r_n}\overline{U}^n_{r_n,m+1}\right) \le C\Delta_n,$$

which can be obtained from (A.23) and (A.26).

This completes the proof of (A.16). To establish (A.17), we show that

$$\mathbb{E}\left(|\beta^n_m|^r \,|\mathcal{H}^n_{mk_n}\right) - \mathbb{E}\left(|\overline{\beta}^n_{r_n,m+1}|^r \,|\mathcal{H}^n_{mk_n}\right) = o_p(1), \tag{A.30}$$

$$\mathbb{E}\left(|\overline{\beta}^n_{r_n,m+1}|^r \,|\mathcal{H}^n_{mk_n}\right) - \mathbb{E}\left(|\overline{\beta}^n_{r_n,m+1}|^r \,|\mathcal{F}^n_{mk_n}\right) = o_p(n^{-1/4}), \tag{A.31}$$

$$\mathbb{E}\left(|\overline{\beta}_{r_n,m+1}|^r \,|\mathcal{F}^n_{mk_n}\right) - \left(\theta\psi_0\sigma^2_{mk_n\Delta_n} + \frac{\psi_1\Sigma_U}{\theta}\right)^{r/2} = o_p(1). \tag{A.32}$$

1. For (A.30), we use the Mean Value Theorem and (A.23) to write

$$\mathbb{E}\left((\beta^n_m)^r - \left(\overline{\beta}^n_{r_n,m+1}\right)^r |\mathcal{H}^n_{mk_n}\right) = \mathbb{E}\left(r\left(\overline{\beta}^n_{r_n,m+1}\right)^{r-1}\left(\overline{\beta}^n_{m,r_n}\right)|\mathcal{H}^n_{mk_n}\right) + o_p(1).$$

Application of the Cauchy-Schwarz inequality yields that the right-hand side is $o_p(1)$ due to (A.25) and Lemma A.2.

2. We now turn to (A.31). For any $l \leq r$, apply (A.3) to write

$$\mathbb{E}\left(\left(n^{1/4}\overline{U}^n_{r_n,m}\right)^l \left|\mathcal{H}^n_{mk_n}\right.\right) = \mathbb{E}\Big(\left(n^{1/4}\overline{U}^n_{r_n,m}\right)^l\Big) + C_{r_n,l}\Lambda_l,$$

with $\mathbb{E}\big(\Lambda_l^2\big) = 1$ and $C_{r_n,l} \leq C r_n^{-v} \leq C n^{-1/4}$ because of (A.18). This means we can replace the conditional moments by the unconditional moments plus a correction term that vanishes asymptotically. Using the notation $C_r^k = \frac{r!}{k!(r-k)!}$ for the binomial coefficients, this gives:

$$\begin{aligned}
&\mathbb{E}\left(\left(\overline{\beta}^n_{r_n,m+1}\right)^r \left|\mathcal{H}^n_{mk_n}\right.\right) \\
&= \mathbb{E}\left(\sum_{k=0}^{r} C_r^k \sigma^k_{mk_n\Delta_n}\left(n^{1/4}\overline{W}^n_{mk_n}\right)^k \left(n^{1/4}\overline{U}^n_{r_n,m}\right)^{r-k} \left|\mathcal{H}^n_{mk_n}\right.\right) \\
&= \sum_{k=0}^{r} C_r^k \sigma^k_{mk_n\Delta_n}\mathbb{E}\left(\left(n^{1/4}\overline{W}^n_{mk_n}\right)^k \left|\mathcal{F}^n_{mk_n}\right.\right)\mathbb{E}\left(\left(n^{1/4}\overline{U}^n_{r_n,m}\right)^{r-k} \left|\mathcal{G}_{mk_n}\right.\right) \\
&= \mathbb{E}\left(\left(\overline{\beta}^n_{r_n,m+1}\right)^r \left|\mathcal{F}^n_{mk_n}\right.\right) + \sum_{k=0}^{r} C_k^r \sigma^k_{mk_n\Delta_n}\mathbb{E}\left(\left(n^{1/4}\overline{W}^n_{mk_n}\right)^k \left|\mathcal{F}^n_{mk_n}\right.\right) C_{r_n,r-k}\Lambda_{r-k}.
\end{aligned}$$

Clearly, the last term is $o_p(1)$ since (A.6) shows that the conditional expectation in the summation is bounded for all k, while $C_{r_n,l} \leq n^{-1/4}$. This proves (A.31).

3. The equality (A.32) is a consequence of the asymptotic distribution of β^n_m, which follows from Lemma A.1, the fact that the sequence of the moments of the noise is uniformly bounded, and the independence of W and U.

This concludes the proof of Lemma A.3. □

Lemma A.4. *Assume that the conditions of Theorem 4.2 hold and let*

$$L_n := n^{-1/4}\sum_{m=0}^{M_n-1}\left(\left(\beta^n_m\right)^2 - \mathbb{E}\left(\left(\beta^n_m\right)^2 \left|\mathcal{H}^n_{mk_n}\right.\right)\right). \tag{A.33}$$

We have the following stable convergence in law:

$$L_n \xrightarrow{\mathcal{L}-s} \sqrt{\frac{2}{\theta}}\int_0^1 \left(\theta\psi_0\sigma_s^2 + \frac{\psi_1\Sigma_U}{\theta}\right)\mathrm{d}W'_s, \tag{A.34}$$

where W' is a standard Wiener process independent of $\mathcal{F}$.

Proof. Let $\vartheta^n_m := n^{-1/4}\left(\left(\beta^n_m\right)^2 - \left(\theta\psi_0\sigma^2_{mk_n\Delta_n} + \frac{\psi_1\Sigma_U}{\theta}\right)\right)$. Then, since $M_n \leq C\sqrt{n}$,

$$L_n = \sum_{m=0}^{M_n-1}\vartheta^n_m + o_p(1),$$

by Lemma A.3. We also have

$$\sum_{m=0}^{M_n-1} \mathbb{E}\left(\vartheta_m^n \left|\mathcal{H}_{mk_n}^n\right.\right) \xrightarrow{\mathbb{P}} 0, \tag{A.35}$$

again by Lemma A.3, and

$$\begin{aligned}\sum_{m=0}^{M_n-1} \mathbb{E}\left((\vartheta_m^n)^2 \left|\mathcal{H}_{mk_n}^n\right.\right) =& \frac{1}{\theta M_n} \sum_{m=0}^{M_n-1} \mathbb{E}\left((\beta_m^n)^4 \left|\mathcal{H}_{mk_n}^n\right.\right) + \frac{1}{\theta M_n} \sum_{m=0}^{M_n-1} \left(\theta\psi_0\sigma_{mk_n\Delta_n}^2 + \frac{\psi_1\Sigma_U}{\theta}\right)^2 \\ &- \frac{2}{\theta M_n} \sum_{m=0}^{M_n-1} \mathbb{E}\left((\beta_m^n)^2 \left|\mathcal{H}_{mk_n}^n\right.\right) \left(\theta\psi_0\sigma_{mk_n\Delta_n}^2 + \frac{\psi_1\Sigma_U}{\theta}\right) + o_p(\Delta_n^{1/4}).\end{aligned}$$

The last remainder term $o_p(\Delta_n^{1/4})$ is due to the approximation $M_n = \sqrt{n}/\theta + o(n^{1/4})$. Now it follows from (A.17) and a Riemann approximation that

$$\sum_{m=0}^{M_n-1} \mathbb{E}\left((\vartheta_m^n)^2 \left|\mathcal{H}_{mk_n}^n\right.\right) \xrightarrow{\mathbb{P}} \frac{2}{\theta} \int_0^1 \left(\theta\psi_0\sigma_u^2 + \frac{\psi_1\Sigma_U}{\theta}\right)^2 \mathrm{d}u. \tag{A.36}$$

Next, denote $\overline{\Delta_m^n Z} = Z_{(m+1)k_n}^n - Z_{mk_n}^n$, for any process Z. We will show that

$$\sum_{m=0}^{M_n-1} \mathbb{E}\left(\vartheta_m^n \overline{\Delta_m^n N} \left|\mathcal{H}_{mk_n}^n\right.\right) \xrightarrow{\mathbb{P}} 0, \tag{A.37}$$

for any bounded martingale N defined on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

According to Jacod et al. (2009) and the proof of Theorem IX 7.28 of Jacod and Shiryaev (2003) it suffices to consider martingales in $\mathcal{N}^0$ or $\mathcal{N}^1$, where $\mathcal{N}^0$ is the set of all bounded martingales on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ which are orthogonal to W, and where $\mathcal{N}^1$ is the set of all martingales having a limit $N_\infty = f(Y_{t_1}, \ldots, Y_{t_q})$, where f is any bounded Borel function on $\mathbb{R}^q$, $t_1 < \ldots < t_q$ and $q \geq 1$.

First, let $N \in \mathcal{N}^0$ and let $\widetilde{\mathcal{F}}'_t = \bigcap_{s>t} \mathcal{F}_s \otimes \mathcal{G}$. Then, for any $t > mk_n\Delta_n$, $\overline{\vartheta}_m^n(t) := \mathbb{E}\left(\vartheta_m^n \left|\widetilde{\mathcal{F}}'_t\right.\right)$, conditional on $\sigma_{mk_n\Delta_n}$, is a martingale with respect to the filtration generated by $\{W_t - W_{mk_n\Delta_n} | t > mk_n\Delta_n\}$. By the martingale representation theorem, we have $\overline{\vartheta}_m^n(t) = \overline{\vartheta}_m^n(mk_n\Delta_n) + \int_{mk_n\Delta_n}^t \tau_u \mathrm{d}W_u$ for some predictable process τ. The orthogonality of W and N and the martingale property of N imply that

$$\mathbb{E}\left(\vartheta_m^n \overline{\Delta_m^n N} \left|\widetilde{\mathcal{F}}'_{mk_n\Delta_n}\right.\right) = \mathbb{E}\left(\left(\vartheta_m^n - \overline{\vartheta}_m^n(mk_n\Delta_n)\right) \overline{\Delta_m^n N} + \overline{\vartheta}_m^n(mk_n\Delta_n) \overline{\Delta_m^n N} \left|\widetilde{\mathcal{F}}'_{mk_n\Delta_n}\right.\right) = 0,$$

which gives

$$\mathbb{E}\left(\vartheta_m^n \overline{\Delta_m^n N} \left|\mathcal{H}_{mk_n}^n\right.\right) = 0, \tag{A.38}$$

since $\mathcal{H}_{mk_n}^n \subset \widetilde{\mathcal{F}}'_{mk_n\Delta_n}$.

Next, assume that $N \in \mathcal{N}^1$. It can be shown (see Jacod et al. (2009)) that there exists some $\hat{f}_t$ such that for all $t \in [t_l, t_{l+1})$, $N_t = \hat{f}_t(Y_{t_0}, Y_{t_1}, \ldots, Y_{t_l}) := \mathbb{E}(f(Y_{t_0}, \ldots, Y_{t_l}, Y_{t_{l+1}}, \ldots, Y_{t_q}) | \mathcal{F}_t)$ with

$t_0 = 0, t_{q+1} = \infty$, and such that it is measurable in $(Y_{t_1}, \ldots, Y_{t_l})$. Hence, $\overline{\Delta_m^n N} = 0$ if it does not cover any of the points $t_1, \ldots, t_{q+1}$. But such intervals (to compute $\overline{\Delta_m^n N}$) that contain any of the points $t_1, \ldots, t_{q+1}$ are at most $q+1$ in number. Furthermore, by the boundedness of N and the conditional Cauchy-Schwarz inequality, we have the following:

$$\mathbb{E}\left(\left|\vartheta_m^n \overline{\Delta_m^n N}\right| \left|\mathcal{H}_{mk_n}^n\right.\right) \leq \sqrt{\mathbb{E}\left((\vartheta_m^n)^2 \left|\mathcal{H}_{mk_n}^n\right.\right)}\sqrt{\mathbb{E}\left(\left(\overline{\Delta_m^n N}\right)^2 \left|\mathcal{H}_{mk_n}^n\right.\right)} = O_p(n^{-1/4}).$$

Now (A.37) follows since there are at most finitely many such intervals.

Due to the fact that ϑ_m^n is an even functional of $n^{1/4}\overline{W}_{mk_n}^n$ and $n^{1/4}\overline{U}_{mk_n}^n$ we know that both have a symmetric asymptotic distribution, and

$$\mathbb{E}\left(\vartheta_m^n \overline{\Delta_m^n W} \left|\mathcal{H}_{mk_n}^n\right.\right) \xrightarrow{\mathbb{P}} 0. \tag{A.39}$$

From (A.17), we deduce that $(\vartheta_m^n)^2 \mathbf{1}_{\{|\vartheta_m^n| > \varepsilon\}} = o_p(n^{-1/2})$ for any $\varepsilon > 0$, so we have

$$\sum_{m=0}^{M_n-1} \mathbb{E}\left((\vartheta_m^n)^2 \mathbf{1}_{\{|\vartheta_m^n| > \varepsilon\}} \left|\mathcal{H}_{mk_n}^n\right.\right) \xrightarrow{\mathbb{P}} 0. \tag{A.40}$$

Now the proof is complete in view of (A.35)-(A.40), and Theorem IX.7.28 of Jacod and Shiryaev (2003). □

Lemma A.5. *Assume that the conditions of Theorem 4.2 hold. We then have that*

$$\sum_{m=0}^{M_n-1} \left(\overline{Y}_{mk_n}^n\right)^2 - \frac{1}{\sqrt{n}} \sum_{m=0}^{M_n-1} (\beta_m^n)^2 = o_p(n^{-1/4}). \tag{A.41}$$

Proof. Denote

$$\widetilde{Y}_m^n = n^{-1/4}\beta_m^n = \sigma_{mk_n\Delta_n} \overline{W}_{mk_n}^n + \overline{U}_{mk_n}^n. \tag{A.42}$$

Then,

$$\mathbb{E}\left(\left|\sum_{m=0}^{M_n-1} \left(\overline{Y}_{mk_n}^n\right)^2 - \frac{1}{\sqrt{n}} \sum_{m=0}^{M_n-1} (\beta_m^n)^2\right|\right) \leq \sum_{m=0}^{M_n-1} \sqrt{\mathbb{E}\left(\left(\overline{Y}_{mk_n}^n - \widetilde{Y}_m^n\right)^2\right)}\sqrt{\mathbb{E}\left(\left(\overline{Y}_{mk_n}^n + \widetilde{Y}_m^n\right)^2\right)}.$$

Since $\sqrt{\mathbb{E}\left(\left(\overline{Y}_{mk_n}^n + \widetilde{Y}_m^n\right)^2\right)} = O(n^{-1/4})$ by (A.6), the result is proven if

$$\sum_{m=0}^{M_n-1} \sqrt{\mathbb{E}\left(\left(\overline{Y}_{mk_n}^n - \widetilde{Y}_m^n\right)^2\right)} \to 0. \tag{A.43}$$

But this follows directly from Lemma 7.8 in Barndorff-Nielsen et al. (2006). □

A.3 Proofs of the Results in Section 3 and Subsection 4.1

A.3.1 Proof of Proposition 3.1

Proof. For any process Z, we write $\Delta^n_{i,j}Z := Z^n_{i+j} - Z^n_i$, for $j = 1, 2, \dots, n-i$. The process Y then satisfies

$$\sum_{i=0}^{n-j}(\Delta^n_{i,j}Y)^2 = \sum_{i=0}^{n-j}(\Delta^n_{i,j}X)^2 + 2\sum_{i=0}^{n-j}\Delta^n_{i,j}X\ \Delta^n_{i,j}U + \sum_{i=0}^{n-j}(\Delta^n_{i,j}U)^2. \tag{A.44}$$

We now analyze the asymptotic properties of the three components on the right-hand side of (A.44):

(i) First note that $\sum_{i=0}^{n-j}(\Delta^n_{i,j}X)^2/j \xrightarrow{\mathbb{P}} [X, X]$, where $[X, X]$ is the quadratic variation of X.

(ii) By the independence of X and U, we have

$$\sum_{i=0}^{n-j}\mathbb{E}\Big(\big(\Delta^n_{i,j}X\ \Delta^n_{i,j}U\big)^2\Big) = \sum_{i=0}^{n-j}\mathbb{E}\Big(\big(\Delta^n_{i,j}X\big)^2\Big)\mathbb{E}\Big(\big(\Delta^n_{i,j}U\big)^2\Big) \le Cj. \tag{A.45}$$

The last inequality follows from the fact that U has bounded moments and from an application of (A.1). Next,

$$\begin{aligned}
&\sum_{i,i':i<i'}\mathbb{E}\big(\Delta^n_{i,j}X\ \Delta^n_{i,j}U\ \Delta^n_{i',j}X\ \Delta^n_{i',j}U\big)\\
=&\sum_{i,i':i<i'}\mathbb{E}\big(\Delta^n_{i,j}X\ \Delta^n_{i',j}X\big)\mathbb{E}\big(\Delta^n_{i,j}U\ \Delta^n_{i',j}U\big)\\
\le& Cj\Delta_n\left(\sum_{i,i':i+j<i'}\mathbb{E}\big(\Delta^n_{i,j}U\ \Delta^n_{i',j}U\big) + \sum_{i,i':i+j\ge i'>i}\mathbb{E}\big(\Delta^n_{i,j}U\ \Delta^n_{i',j}U\big)\right)\\
\le& Cj^2.
\end{aligned} \tag{A.46}$$

The first inequality follows from the Cauchy-Schwarz inequality and (A.1). To establish the second inequality, we apply the Cauchy-Schwarz inequality, (A.5), and the fact that $v > 1$, to obtain

$$\begin{aligned}
\sum_{i,i':i+j<i'}\mathbb{E}\big(\Delta^n_{i,j}U\ \Delta^n_{i',j}U\big) &= \sum_{i,i':i+j<i'}\mathbb{E}\big(\Delta^n_{i,j}U\ \mathbb{E}\left(\Delta^n_{i',j}U\,|\mathcal{F}_{(i+j)\Delta_n}\right)\big)\\
&\le C\sum_i\sum_{i':i+j<i'}\sqrt{\mathbb{E}\Big(\Big(\mathbb{E}\left(\Delta^n_{i',j}U\,|\mathcal{F}_{(i+j)\Delta_n}\right)\Big)^2\Big)}\\
&\le C\sum_i\sum_{i':i+j<i'}(i'-(i+j))^{-v} \le C\Delta_n^{-1}.
\end{aligned} \tag{A.47}$$

Equations (A.45) and (A.46) imply that $\mathbb{E}\Big(\Big(\sum_{i=0}^{n-j}\Delta^n_{i,j}X\ \Delta^n_{i,j}U\Big)^2\Big) \le Cj^2$, so

$$\sum_{i=0}^{n-j}\Delta^n_{i,j}X\ \Delta^n_{i,j}U = O_p(j). \tag{A.48}$$

(iii) Turning to the last sum of (A.44), let $\nu_j := \mathbb{E}\big((U^n_{i+j} - U^n_i)^2\big) = 2(\gamma(0) - \gamma(j))$. For $i > j$, we obtain the following, using similar arguments as the ones used to prove (A.47):

$$\left|\mathbf{Cov}\left((U_j^n - U_0^n)^2, (U_{i+j}^n - U_i^n)^2\right)\right| \leq C(i-j)^{-v},$$

which implies

$$\mathbb{E}\left(\left(\sum_{i=0}^{n-j}\left((\Delta_{i,j}^n U)^2 - \nu_j\right)\right)^2\right) \leq C\Delta_n^{-1} j. \tag{A.49}$$

For any fixed j and any j_n satisfying $\Delta_n j_n \to 0, j_n \to \infty$, we have by (A.48), (A.49) and (4) that

$$\begin{aligned}
&\widehat{\langle Y, Y\rangle}(j)_n - (\gamma(0) - \gamma(j)) = O_p\left(\sqrt{\Delta_n j}\right); \\
&\widehat{\langle Y, Y\rangle}(j_n)_n - \gamma(0) = O_p\left(\max\left\{\sqrt{\Delta_n j_n}, j_n^{-v}\right\}\right).
\end{aligned} \tag{A.50}$$

Now the stated result follows. □

A.3.2 Proof of Proposition 3.2

Proof. By Itô's isometry, we have

$$\begin{aligned}
\mathbb{E}_\sigma\left(\sum_{i=0}^{n-j}(\Delta_{i,j}^n X)^2\right) &= \sum_{i=0}^{n-j}\sum_{h=i}^{i+j-1}\int_{h\Delta n}^{(h+1)\Delta_n}\sigma_s^2 \mathrm{d}s \;=\; \sum_{h=0}^{n-1}\sum_{i=0\vee(h-j+1)}^{(n-j)\wedge h}\int_{h\Delta n}^{(h+1)\Delta_n}\sigma_s^2 \mathrm{d}s \\
&= \sum_{h=j-1}^{n-j}\sum_{i=h-j+1}^{h}\int_{h\Delta n}^{(h+1)\Delta_n}\sigma_s^2 \mathrm{d}s + o(j^2\Delta_n) \;=\; j\int_{(j-1)\Delta_n}^{(n+1-j)\Delta_n}\sigma_s^2 \mathrm{d}s + o(j^2\Delta_n),
\end{aligned}$$

where we reversed the order of summation in the second equality, while the stochastic orders follow from the regularity conditions on the volatility path around 0 and 1. Hence, we have

$$2(n-j+1)\mathbb{E}_\sigma\left(\widehat{\langle X, X\rangle}(j)_n\right) = j\int_0^1 \sigma_s^2 \mathrm{d}s + O_p(j^2\Delta_n).$$

Furthermore, it is immediate that $\mathbb{E}_\sigma\left(\sum_{i=0}^{n-j}(\Delta_{i,j}^n U)^2\right) = 2(n-j+1)(\gamma(0)-\gamma(j))$. Thus, we have, by the independence of X and U,

$$\mathbb{E}_\sigma\left(\widehat{\langle Y, Y\rangle}(j)_n\right) = \frac{j\int_0^1 \sigma_s^2 \mathrm{d}s}{2(n-j+1)} + \gamma(0) - \gamma(j) + O_p(j^2\Delta_n^2).$$

□

A.3.3 Proof of Proposition 3.3

Proof. We note that $|\Sigma_U - \widehat{\Sigma}_{U_n}|$ is smaller than

$$2\sum_{j=0}^{\ell_n}|\gamma(j) - \widehat{\gamma(j)}_n| + 2\sum_{j=\ell_n+1}^{\infty}|\gamma(j)|.$$

The last sum is of order $(\ell_n)^{1-v}$ with $v > 3$ and $\ell_n \geq Cn^{1/8}$ by (14), so it is $o(n^{-1/4})$. For the first sum we use definitions (6)-(8) and then apply (A.50) to conclude that for $j \leq \ell_n$:

$$|\gamma(j) - \widehat{\gamma(j)}_n| = |\gamma(j) - \widehat{\langle Y, Y\rangle}(j_n)_n + \widehat{\langle Y, Y\rangle}(j)_n| = O_p\left(\max\left\{\sqrt{\Delta_n j_n}, j_n^{-v}, \sqrt{\Delta_n \ell_n}\right\}\right).$$

Our restrictions in (14) then guarantee that $|\gamma(j) - \widehat{\gamma(j)}_n| = O_p((\Delta_n)^{5/12})$ while $\ell_n = o((\Delta_n)^{-1/6})$, so

$$\sum_{j=0}^{\ell_n} |\gamma(j) - \widehat{\gamma(j)}_n| = o_p(n^{-1/4}); \qquad |\Sigma_U - \widehat{\Sigma}_{U_n}| = o_p(n^{-1/4}). \tag{A.51}$$

This establishes the result. □

A.3.4 Proof of Theorem 4.1

Proof. We present the proof in three steps, which correspond to the following three equations:

$$\mathrm{PAV}(Y, r)_n - \frac{1}{M_n}\mathrm{PAV}^n \xrightarrow{\mathbb{P}} 0, \tag{A.52}$$

$$\frac{1}{M_n}\mathrm{PAV}^n - \frac{1}{M_n}\widetilde{\mathrm{PAV}}^n \xrightarrow{\mathbb{P}} 0, \tag{A.53}$$

$$\frac{1}{M_n}\widetilde{\mathrm{PAV}^n} - \mathrm{PAV}(Y, r) \xrightarrow{\mathbb{P}} 0. \tag{A.54}$$

We invoke Lemmas A.2 and A.3, which in turn rely on Lemma A.1.

(i) To prove (A.52), recall our choice[2] of $M_n = \left\lfloor \frac{n}{k_n} \right\rfloor$. The difference on the left-hand side of (A.52) is a sum of martingale differences:

$$\begin{aligned}&\mathrm{PAV}(Y, r)_n - \frac{1}{M_n}\mathrm{PAV}^n \\ &= \sum_{m=0}^{M_n - 1} \frac{1}{\sqrt{n}} \left(\left| n^{\frac{1}{4}} \overline{Y}^n_{mk_n} \right|^r - \mathbb{E}\left(\left| n^{\frac{1}{4}} \overline{Y}^n_{mk_n} \right|^r |\mathcal{H}^n_{mk_n} \right) \right).\end{aligned}$$

In light of Lemma 2.2.11 in Jacod and Protter (2011), it suffices to show that

$$\frac{1}{n} \sum_{m=0}^{M_n - 1} \mathbb{E}\left(\left| n^{\frac{1}{4}} \overline{Y}^n_{mk_n} \right|^{2r} |\mathcal{H}^n_{mk_n} \right) \xrightarrow{\mathbb{P}} 0. \tag{A.55}$$

But this follows from the boundedness established in Lemma A.2 and the choice of M_n.

(ii) To establish (A.53), we proceed in several steps:

(a) We first note that the error when approximating $n^{1/4}\overline{Y}^n_i$ by β^n_m, denoted by ξ^n_m, is small in the sense that

$$\frac{1}{M_n} \sum_{m=0}^{M_n - 1} \mathbb{E}\left(|\xi^n_m|^2 \right) \to 0. \tag{A.56}$$

[2] We interchangeably use $k_n \Delta_n$ and $1/M_n$ in the sequel; the difference of the two is always negligible.

To see this, we write

$$\xi_m^n = n^{1/4} \left(\int_{mk_n\Delta_n}^{(m+1)k_n\Delta_n} b_s G_{mk_n}^n(s) \mathrm{d}s + \int_{mk_n\Delta_n}^{(m+1)k_n\Delta_n} \left(\sigma_s - \sigma_{mk_n\Delta_n}\right) G_{mk_n}^n(s) \mathrm{d}W_s \right).$$

Since b and G are bounded, we have

$$\mathbb{E}\left(n^{1/2} \left(\int_{mk_n\Delta_n}^{(m+1)k_n\Delta_n} b_s G_{mk_n}^n(s) \mathrm{d}s \right)^2 \right) \leq C n^{1/2} (k_n\Delta_n)^2 \leq C\sqrt{\Delta_n}.$$

By Itô isometry,

$$\begin{aligned} &\mathbb{E}\left(n^{1/2} \left(\int_{mk_n\Delta_n}^{(m+1)k_n\Delta_n} \left(\sigma_s - \sigma_{mk_n\Delta_n}\right) G_{mk_n}^n(s) \mathrm{d}W_s \right)^2 \right) \\ &\qquad \leq \; C\Delta_n^{-1/2} \int_{mk_n\Delta_n}^{(m+1)k_n\Delta_n} \mathbb{E}\Big(\left(\sigma_s - \sigma_{mk_n\Delta_n}\right)^2 \Big) \mathrm{d}s, \end{aligned}$$

and hence

$$\frac{1}{M_n} \sum_{m=0}^{M_n-1} \mathbb{E}\Big(|\xi_m^n|^2 \Big) \leq C \left(\Delta_n^{1/2} + \int_0^1 \mathbb{E}\bigg(\Big(\sigma_s - \sigma_{k_n\Delta_n \lfloor \frac{s}{k_n\Delta_n} \rfloor} \Big)^2 \bigg) \mathrm{d}s \right) \to 0,$$

by Lebesgue's dominated convergence theorem, since $\sigma_{k_n\Delta_n \lfloor \frac{s}{k_n\Delta_n} \rfloor} \to \sigma_s$ and σ is bounded.

(b) Next, define the approximation error

$$\zeta_m^n := \frac{\left| n^{1/4} \overline{Y}_{mk_n}^n \right|^r - |\beta_m^n|^r}{\theta}.$$

We note that this error is also small:

$$\frac{1}{M_n} \sum_{m=0}^{M_n-1} \mathbb{E}(|\zeta_m^n|) \to 0, \tag{A.57}$$

which follows from

$$\frac{1}{M_n} \sum_{m=0}^{M_n-1} \mathbb{E}\Big(|\zeta_m^n|^2 \Big) \to 0. \tag{A.58}$$

This can be proved using similar arguments as in the proof of (A.56). Equation (A.57) then follows, and it implies

$$\frac{1}{M_n} \sum_{m=0}^{M_n-1} \mathbb{E}\left(\zeta_m^n \,\middle|\, \mathcal{H}_{mk_n}^n \right) \xrightarrow{\mathbb{P}} 0, \tag{A.59}$$

by the Markov inequality.

(c) By Lemma A.3 we have

$$\mathbb{E}\left(|\beta_m^n|^r \,\middle|\, \mathcal{H}_{mk_n}^n \right) = \mu_r \left(\theta\psi_0 \sigma_{mk_n\Delta_n}^2 + \frac{\psi_1 \Sigma_U}{\theta} \right)^{\frac{r}{2}} + o_p(1), \tag{A.60}$$

which holds uniformly in m for any even integer $r \geq 2$. Now (A.53) follows from (A.59) and (A.60).

(iii) Following Proposition 2.2.8 in Jacod and Protter (2011), we see that (A.54) boils down to convergence of a Riemann approximation.

This finishes the proof of Theorem 4.1. $\square$

A.3.5 Proof of Theorem 4.2

Proof. We have, by the definition of $\widehat{\mathrm{IV}}_n$ and (A.16) of Lemma A.3 that

$$n^{1/4}\widehat{\mathrm{IV}}_n = n^{1/4}(\psi_0)^{-1}\left(\sum_{m=0}^{M_n-1}|\overline{Y}^n_{mk_n}|^2 - \psi_1\theta^{-2}\widehat{\Sigma}_{U_n}\right),$$
$$n^{1/4}\mathrm{IV} = n^{1/4}(\psi_0)^{-1}\left(\frac{1}{\theta M_n}\sum_{m=0}^{M_n-1}\mathbb{E}\left((\beta^n_m)^2\,|\mathcal{H}^n_{mk_n}\right) - \psi_1\theta^{-2}\Sigma_U\right) + o_p(1).$$

Subtraction gives, due to (A.41) of Lemma A.5 and because $\theta M_n = \sqrt{n}$, that $n^{1/4}(\widehat{\mathrm{IV}}_n - \mathrm{IV})$ equals

$$(\psi_0)^{-1}L_n + Cn^{1/4}(\Sigma_U - \widehat{\Sigma}_{U_n}) + o_p(1),$$

with L_n as defined in (A.33) of Lemma A.4. The first statement of Theorem 4.2 now follows from that Lemma and (A.51), while the second statement is implied by the consistency result in (21). $\square$

A.4 Proofs of the Results in Subsection 4.2

In this subsection we first establish several lemmas to facilitate the proofs of our results in Subsection 4.2. We follow the classical approach in Jacod et al. (2009) and also use several estimates that have been derived in Jacod et al. (2019). Our proofs are often less involved than those in the last paper. This is partly due to our Lemma A.1, which we proved under relatively mild assumptions and from which the higher order moments of the pre-averaged noise process can be easily obtained. Moreover, our setting is not as general as in Jacod et al. (2019).

A.4.1 Auxiliary Lemmas for Subsection 4.2

In the following Lemmas A.6 to A.10, we assume the conditions of Theorem 4.3 are satisfied.

Lemma A.6. *For any $q \geq 1$, we have*

$$\left|\mathbb{E}\left(\widehat{X}^n_i\,|\mathcal{F}^n_i\right)\right| \leq C\Delta_n; \quad \mathbb{E}\left(\left|\widehat{X}^n_i\right|^q|\mathcal{F}^n_i\right) \leq C_q\Delta_n^{q/2}. \tag{A.61}$$

Proof. Using the decomposition in the proof of Lemma 5.2 in Jacod et al. (2009), we have by Itô's formula that $\frac{\widehat{X}_i^n}{2} = \sum_{\ell=1}^3 D_{i,\ell}^n$, where

$$D_{i,1}^n = \int_{i\Delta_n}^{(i+k_n-1)\Delta_n} X_i^n(t)\mathrm{d}M_i^n(t), \quad D_{i,2}^n = b_{i\Delta_n}\int_{i\Delta_n}^{(i+k_n-1)\Delta_n} M_i^n(t)G_i^n(t)\mathrm{d}t,$$
$$D_{i,3}^n = \int_{i\Delta_n}^{(i+k_n-1)\Delta_n} M_i^n(t)(b_t - b_{i\Delta_n})G_i^n(t)\mathrm{d}t + \int_{i\Delta_n}^{(i+k_n-1)\Delta_n} B_i^n(t)\mathrm{d}B_i^n(t).$$

The boundedness of b, σ and g imply that we have that $\mathbb{E}\left(|M_i^n(t)|^q \,|\mathcal{F}_i^n\right) \le C(k_n\Delta_n)^{q/2}$ and that $\mathbb{E}\left(|B_i^n(t)|^q \,|\mathcal{F}_i^n\right) \le C(k_n\Delta_n)^q$, and since $k_n = \theta\Delta_n^{-1/2} + o\left(\Delta_n^{-1/4}\right)$ this gives

$$\mathbb{E}\left(\left|D_{i,2}^n\right|^q |\mathcal{F}_i^n\right) \le C_q\Delta_n^{3q/4}; \tag{A.62}$$
$$\mathbb{E}\left(\left|D_{i,3}^n\right|^q |\mathcal{F}_i^n\right) \le C_q\Delta_n^{q}. \tag{A.63}$$

The boundedness of σ and g also establish that $|\mathbb{E}\left(M_i^n(t)\,|\mathcal{F}_i^n\right)| = 0$ which gives, together with the boundedness of b and g, that

$$\left|\mathbb{E}\left(D_{i,2}^n\,|\mathcal{F}_i^n\right)\right| \le C\Delta_n. \tag{A.64}$$

The martingale property of M yields $\mathbb{E}\left(D_{i,1}^n\,|\mathcal{F}_i^n\right) = \mathbb{E}\left(\int_{i\Delta_n}^{(i+k_n-1)\Delta_n} X_i^n(t)\mathrm{d}M_i^n(t)\,|\mathcal{F}_i^n\right) = 0$ and combining this with (A.63) and (A.64) proves the first part of (A.61). The second part of (A.61) follows from (A.62), (A.63) and

$$\mathbb{E}\left(\left|D_{i,1}^n\right|^q |\mathcal{F}_i^n\right) \le C_q\Delta_n^{q/2},$$

which can be obtained by applying the Burkholder-Davis-Gundy inequalities. This finishes the proof. □

Lemma A.7. *For any $p \ge 2$, we have*

$$\mathbb{E}\Big(\mathbb{E}\left((\zeta(p)_i^n)^4\,|\mathcal{K}_i^n\right)\Big) \le C_p; \tag{A.65}$$
$$\mathbb{E}\Big(\left(\mathbb{E}\left(\zeta(p)_i^n\,|\mathcal{K}_i^n\right)\right)^2\Big) \le C_p\Delta_n. \tag{A.66}$$

Proof. We have by Lemma A.1 that

$$\mathbb{E}\Big(\left(\overline{U}_i^n\right)^8\Big) \le C\Delta_n^2, \quad \mathbb{E}\Big(\left(\overline{U}_i^n\right)^4\Big) \le C\Delta_n, \quad \alpha_n = \mathbb{E}\Big(\left(\overline{U}_i^n\right)^2\Big) \le C\Delta_n^{1/2}. \tag{A.67}$$

This implies

$$\mathbb{E}\Big(\mathbb{E}\left(\left(\widehat{U}_i^n\right)^4 |\mathcal{K}_i^n\right)\Big) \le C\left(\mathbb{E}\Big(\left(\overline{U}_i^n\right)^8\Big) + \alpha_n^4\right) \le C\Delta_n^2. \tag{A.68}$$

Hölder's inequality gives $\left((\zeta(p)_j^n)^4\right) \le C_p k_n^3 \sum_{j=i}^{i+pk_n-1}\left(\left(\widehat{X}_i^n\right)^4 + \left(\widehat{U}_i^n\right)^4 + 2\left(\overline{U}_i^n\right)^4\left(\overline{X}_i^n\right)^4\right)$. Now (A.67), (A.68), together with the second part of (A.61), the independence of X and U, and (A.6) yield (A.65).

We now turn to (A.66). By (A.4) we have

$$\sum_{j=i}^{i+pk_n-1}\mathbb{E}\Big(\Big(\mathbb{E}\left(\widehat{U}_j^n\left|\mathcal{G}_{i-\lfloor\frac{k_n}{2}\rfloor}\right.\right)\Big)^2\Big)\leq\sum_{j=i}^{i+pk_n-1}\frac{C}{(j-i+\lfloor k_n/2\rfloor)^{2v}}\leq\frac{C}{k_n^{2v-1}}\leq C(\Delta_n)^{v-\frac{1}{2}}. \tag{A.69}$$

On the other hand, for any $i\leq j\leq i+pk_n-1$ we have, by again applying (A.4) and using $|\overline{g}_k^n|\leq C/k_n$, that

$$\begin{aligned}\mathbb{E}\Big(\Big(\mathbb{E}\left(\overline{U}_j^n\left|\mathcal{G}_{i-\lfloor\frac{k_n}{2}\rfloor}\right.\right)\Big)^2\Big)&=\mathbb{E}\left(\left(\sum_{k=0}^{k_n-1}\overline{g}_k^n\mathbb{E}\left(U_{j+k}^n\left|\mathcal{G}_{i-\lfloor\frac{k_n}{2}\rfloor}\right.\right)\right)^2\right)\\&\leq k_n(C/k_n)^2\sum_{k=0}^{k_n-1}\mathbb{E}\Big(\Big(\mathbb{E}\left(U_{j+k}^n\left|\mathcal{G}_{i-\lfloor\frac{k_n}{2}\rfloor}\right.\right)\Big)^2\Big)\\&\leq(C/k_n)\sum_{k=0}^{k_n-1}\frac{1}{\left(j+k-i+\lfloor\frac{k_n}{2}\rfloor\right)^{2v}}\leq\frac{C}{k_n\left(j-i+\lfloor\frac{k_n}{2}\rfloor\right)^{2v-1}},\end{aligned}$$

whence

$$\sum_{j=i}^{i+pk_n-1}\mathbb{E}\Big(\Big(\mathbb{E}\left(\overline{U}_j^n\left|\mathcal{G}_{i-\lfloor\frac{k_n}{2}\rfloor}\right.\right)\Big)^2\Big)\leq\frac{C}{k_n^{2v-1}}\leq C(\Delta_n)^{v-1/2}. \tag{A.70}$$

By the independence of X and U, (A.6) and (A.61), we deduce

$$\begin{aligned}\mathbb{E}\left(\zeta(p)_i^n\left|\mathcal{K}_i^n\right.\right)&=\sum_{j=i}^{i+pk_n-1}\mathbb{E}\left(\widehat{X}_i^n+\widehat{U}_i^n+2\overline{U}_i^n\overline{X}_i^n\left|\mathcal{K}_i^n\right.\right)\\&\leq C_p\Delta_n^{1/2}+\sum_{j=i}^{i+pk_n-1}\mathbb{E}\left(\widehat{U}_j^n\left|\mathcal{G}_{i-\lfloor\frac{k_n}{2}\rfloor}\right.\right)+2\Delta_n^{1/2}\sum_{j=i}^{i+pk_n-1}\mathbb{E}\left(\overline{U}_j^n\left|\mathcal{G}_{i-\lfloor\frac{k_n}{2}\rfloor}\right.\right).\end{aligned} \tag{A.71}$$

Since $v>2$, we can now apply Hölder's inequality to the square of this expression to get (A.66) from (A.69), (A.70) and (A.71). □

Lemma A.8. *Let $t_{j,p}^n=j(p+1)k_n\Delta_n$ and define $\Xi_{ij}=-\int_0^1 s\phi_i(s)\phi_j(s)\mathrm{d}s$, and $\Lambda_{ij}(p)=p\Phi_{ij}+\Xi_{ij}$ for $i,j\in\{0,1\}$. We then have*

$$\mathbb{E}\left(\left|\mathbb{E}\left(\left(\eta(p)_j^n\right)^2-\frac{4k_n^2\Delta_n^2\sigma^4_{t_{j,p}^n}}{\psi_0^2}\Lambda_{00}(p)-\frac{4\Delta_n\Sigma_U^2}{\theta^2\psi_0^2}\Lambda_{11}(p)-\frac{8\Delta_n\sigma^2_{t_{j,p}^n}\Sigma_U}{\psi_0^2}\Lambda_{01}(p)\left|\mathcal{J}(p)_j^n\right.\right)\right|\right)\leq C_p\Delta_n^{5/4}. \tag{A.72}$$

Proof. First, we note that

$$\begin{aligned}(\zeta(p)_i^n)^2=\sum_{j,j'=0}^{pk_n-1}(&\widehat{X}_{i+j}^n\widehat{X}_{i+j'}^n+\widehat{U}_{i+j}^n\widehat{U}_{i+j'}^n+\widehat{X}_{i+j}^n\widehat{U}_{i+j'}^n+\widehat{U}_{i+j}^n\widehat{X}_{i+j'}^n+4\overline{X}_{i+j}^n\overline{U}_{i+j}^n\overline{X}_{i+j'}^n\overline{U}_{i+j'}^n\\&+2\widehat{X}_{i+j}^n\overline{X}_{i+j'}^n\overline{U}_{i+j'}^n+2\overline{X}_{i+j}^n\overline{U}_{i+j}^n\widehat{X}_{i+j'}^n+2\widehat{U}_{i+j}^n\overline{X}_{i+j'}^n\overline{U}_{i+j'}^n+2\overline{X}_{i+j}^n\overline{U}_{i+j}^n\widehat{U}_{i+j'}^n).\end{aligned}$$

Applying the estimate (A.20) in Jacod et al. (2019), we get

$$\mathbb{E}\left(\left|\sum_{j,j'=0}^{pk_n-1}\mathbb{E}\left(\widehat{X}^n_{i+j}\widehat{X}^n_{i+j'}-4(\sigma^n_i)^4\overline{G}^n_i(j,j')\,|\mathcal{F}^n_i\right)\right|\right)\le C_p\Delta_n^{1/4}. \tag{A.73}$$

Another estimate gives (see the proof of Lemma A.5 in Jacod et al. (2019)):

$$\left|\mathbb{E}\left(\overline{X}^n_{i+j}\overline{X}^n_{i+j'}-(\sigma^n_i)^2\widehat{G}^n_i(j,j')\,|\mathcal{F}^n_i\right)\right|\le C\Delta_n^{3/4},$$

which yields, since X and U are independent while $\mathbb{E}\Big(\overline{U}^n_{i+j}\overline{U}^n_{i+j'}\Big)\le C\Delta_n^{1/2}$, that

$$\mathbb{E}\left(\left|\sum_{j,j'=0}^{pk_n-1}\mathbb{E}\left(\overline{X}^n_{i+j}\overline{U}^n_{i+j}\overline{X}^n_{i+j'}\overline{U}^n_{i+j'}-(\sigma^n_i)^2\widehat{G}^n_i(j,j')\mathbb{E}\Big(\overline{U}^n_{i+j}\overline{U}^n_{i+j'}\Big)\,|\mathcal{K}^n_i\right)\right|\right)\le C_p\Delta_n^{1/4}. \tag{A.74}$$

A direct application of the JS-Lemma then leads to

$$\mathbb{E}\left(\sum_{j,j'=0}^{pk_n-1}\left|\mathbb{E}\left(\widehat{U}^n_{i+j}\widehat{U}^n_{i+j'}\,\Big|\mathcal{G}_{i-\lfloor\frac{k_n}{2}\rfloor}\right)-\mathbb{E}\Big(\widehat{U}^n_{i+j}\widehat{U}^n_{i+j'}\Big)\right|\right)\le C_pk_n^{-(v-2)}\le C_p\Delta_n^{1/4}, \tag{A.75}$$

since $v>5/2$.

We now find bounds for the six remaining terms in the conditional expectation of $(\zeta(p)^n_i)^2$ using symmetry. We first apply the JS-Lemma (A.5) to derive

$$\mathbb{E}\left(\mathbb{E}\left(\widehat{U}^n_{i+j}\,\Big|\mathcal{G}_{i-\lfloor\frac{k_n}{2}\rfloor}\right)^2\right)\le C\mathbb{E}\left(\left(\widehat{U}^n_{i+j}\right)^2\right)(j+k_n/2)^{-2v}\le C\Delta_n(j+k_n/2)^{-2v}, \tag{A.76}$$

$$\mathbb{E}\left(\mathbb{E}\left(\overline{U}^n_{i+j}\,\Big|\mathcal{G}_{i-\lfloor\frac{k_n}{2}\rfloor}\right)^2\right)\le C\mathbb{E}\left(\left(\overline{U}^n_{i+j}\right)^2\right)(j+k_n/2)^{-2v}\le C\sqrt{\Delta_n}(j+k_n/2)^{-2v}. \tag{A.77}$$

We use this to find, by the independence of X and U and using (A.61), that

$$\begin{aligned}\mathbb{E}\left(\left|\mathbb{E}\left(\sum_{j,j'=0}^{pk_n-1}\widehat{X}^n_{i+j}\widehat{U}^n_{i+j'}\,|\mathcal{K}^n_i\right)\right|\right)&\le Cpk_n\Delta_n\sum_{j=0}^{pk_n-1}\mathbb{E}\Big(\Big|\mathbb{E}\left(\widehat{U}^n_{i+j}\,\Big|\mathcal{G}_{i-\lfloor\frac{k_n}{2}\rfloor}\right)\Big|\Big)\\&\le C_p\sqrt{\Delta_n}\sum_{j=0}^{pk_n-1}\sqrt{\Delta_n(j+k_n/2)^{-2v}}\le C_p\Delta_n^{(v+1)/2}.\end{aligned} \tag{A.78}$$

For a second estimate, we apply the Cauchy-Schwarz inequality, the bounds of (A.6) and (A.61) and the independence of X and U to derive

$$\begin{aligned}\mathbb{E}\left(\widehat{X}^n_{i+j}\overline{X}^n_{i+j'}\overline{U}^n_{i+j'}\,|\mathcal{K}^n_i\right)&\le\mathbb{E}\left((\widehat{X}^n_{i+j})^2\,|\mathcal{K}^n_i\right)^{1/2}\mathbb{E}\left((\overline{X}^n_{i+j'}\overline{U}^n_{i+j'})^2\,|\mathcal{K}^n_i\right)^{1/2}\\&\le C\Delta_n^{1/2}\mathbb{E}\left((\overline{X}^n_{i+j'})^2\,|\mathcal{K}^n_i\right)^{1/2}\mathbb{E}\left((\overline{U}^n_{i+j'})^2\,|\mathcal{K}^n_i\right)^{1/2}\\&\le C\Delta_n^{3/4}\mathbb{E}\left((\overline{U}^n_{i+j'})^2\,|\mathcal{K}^n_i\right)^{1/2}.\end{aligned}$$

Therefore, using (A.77),

$$\mathbb{E}\left(\left|\mathbb{E}\left(\sum_{j,j'=0}^{pk_n-1}\widehat{X}_{i+j}^n\overline{X}_{i+j'}^n\overline{U}_{i+j'}^n\,|\mathcal{K}_i^n\right)\right|\right)\leq C\Delta_n^{3/4}pk_n\sum_{j'=0}^{pk_n-1}\sqrt{\sqrt{\Delta_n}(j'+k_n/2)^{-2v}}\leq C\Delta_n^{v/2}. \quad \text{(A.79)}$$

For a third estimate, we use that we know from (A.67) and (A.68) that $n^{1/2}\widehat{U}_{i+j}^n$ and $n^{1/4}\overline{U}_{i+j}^n$ are sequences of stochastic variables with variances that converge to one. Together with the estimates in (A.6), this gives

$$\begin{aligned}\mathbb{E}\left(\left|\mathbb{E}\left(\sum_{j,j'=0}^{pk_n-1}\widehat{U}_{i+j}^n\overline{X}_{i+j'}^n\overline{U}_{i+j'}^n\,|\mathcal{K}_i^n\right)\right|\right)&\leq C\Delta_n^{1/2}\sum_{j,j'=0}^{pk_n-1}\Delta_n^{3/4}\mathbb{E}\left(\left|\mathbb{E}\left(\frac{\overline{U}_{i+j'}^n}{\Delta_n^{1/4}}\frac{\widehat{U}_{i+j}^n}{\Delta_n^{1/2}}\,\Big|\mathcal{G}_{i-\lfloor\frac{k_n}{2}\rfloor}\right)\right|\right)\\&\leq C_p\Delta_n^{5/4}k_n\leq C_p\Delta_n^{3/4}. \quad \text{(A.80)}\end{aligned}$$

Lemmas A.9 and A.10 in Jacod et al. (2019) yield

$$\begin{aligned}\left|\sum_{j,j'=0}^{pk_n-1}\overline{G}_i^n(j,j')-k_n^4\Delta_n^2\Lambda_{00}(p)\right|&\leq C_p\Delta_n^{1/2};\\ \left|\sum_{j,j'=0}^{pk_n-1}\widehat{G}_i^n(j,j')\mathbb{E}\Big(\overline{U}_{i+j}^n\overline{U}_{i+j'}^n\Big)-2k_n^2\Delta_n\Lambda_{01}(p)\Sigma_U\right|&\leq C_p\Delta_n^{1/2}; \quad \text{(A.81)}\\ \left|\sum_{j,j'=0}^{pk_n-1}\mathbb{E}\Big(\widehat{U}_{i+j}^n\widehat{U}_{i+j'}^n\Big)-4\Lambda_{11}(p)\Sigma_U^2\right|&\leq C_p\Delta_n^{1/2}.\end{aligned}$$

Now the result follows from (A.73)-(A.81). □

Lemma A.9. *For any $p\geq 2$, we have*

$$\Delta_n^{-1/4}F(p)_n\xrightarrow{\mathbb{P}}0; \quad \text{(A.82)}$$
$$\Delta_n^{-1/4}F'(p)_n\xrightarrow{\mathbb{P}}0; \quad \text{(A.83)}$$
$$\Delta_n^{-1/4}\widehat{C}(p)_n\xrightarrow{\mathbb{P}}0; \quad \text{(A.84)}$$
$$\Delta_n^{-1/4}\widehat{C}'(p)_n\xrightarrow{\mathbb{P}}0; \quad \text{(A.85)}$$
$$\Delta_n^{-1/4}\widehat{C}''_n\xrightarrow{\mathbb{P}}0; \quad \text{(A.86)}$$
$$\mathbb{E}\left(\sup_{t\leq T}\left(M'(p)_n\right)^2\right)\leq C\sqrt{\Delta_n}/p. \quad \text{(A.87)}$$

Proof. We prove these equations in a number of separate steps.

(1) Proof of (A.82) and (A.83). First, we note that due to (A.76) we have $\mathbb{E}\Big(\Big|\mathbb{E}\left(\widehat{U}_i^n\,|\mathcal{K}_i^n\right)\Big|\Big)\leq Ck_n^{-v}\sqrt{\Delta_n}$. Together with (A.61), and the independence of X and U, we get

$$\mathbb{E}(|\mathbb{E}\left(\Psi_i^n\,|\mathcal{K}_i^n\right)|)\leq C\Delta_n,\quad \mathbb{E}(|\mathbb{E}\left(\zeta(p)_i^n\,|\mathcal{K}_i^n\right)|)\leq Cpk_n\Delta_n.$$

Since $K_n^p \leq \frac{C}{pk_n\Delta_n}$, we have $\mathbb{E}(|F(p)_n|) \leq C\Delta_n^{\frac{1}{2}}$. The same result holds for $F'(p)_n$. Now (A.82) and (A.83) follow.

(2) Proof of (A.84). From the estimates (A.61), (A.6) and (A.67), we have

$$\mathbb{E}\Big((\Psi_i^n)^2\Big) \leq C\Delta_n. \tag{A.88}$$

Since $n - k_n - I_n^p \leq C_p/\sqrt{\Delta_n}$, the claim follows.

(3) Proof of (A.85). Let $\Gamma_n = \sum_{|\ell|\leq \ell_n} \gamma(\ell)$. Then (18) implies

$$\alpha_n - \frac{\psi_1 \Gamma_n}{\theta\sqrt{\Delta_n}(n-k_n+2)} = \alpha_n - \frac{\psi_1\Gamma_n}{k_n} + o(\Delta_n^{\frac{3}{4}}). \tag{A.89}$$

Since $\alpha_n = \frac{1}{k_n}\sum_{|\ell|\leq k_n-1}\phi_1^n(\ell)\gamma(\ell)$, we have

$$\begin{aligned}
\left|\alpha_n - \frac{\psi_1\Gamma_n}{k_n}\right| &= \frac{1}{k_n}\left|\sum_{|\ell|\leq k_n-1}\phi_1^n(\ell)\gamma(\ell) - \psi_1\sum_{|\ell|\leq \ell_n}\gamma(\ell)\right| \\
&\leq \frac{1}{k_n}\left|\psi_1 \sum_{\ell_n<|\ell|\leq k_n-1}\gamma(\ell)\right| + \frac{1}{k_n}\left|\sum_{|\ell|\leq k_n-1}\gamma(\ell)\left(\phi_1^n(\ell)-\psi_1\right)\right| \\
&\leq \frac{C}{k_n\ell_n^{v-1}} + \frac{C}{k_n}\sum_{|\ell|\leq k_n-1}\frac{\gamma(\ell)\ell}{k_n} \leq C\Delta_n^{1\wedge\left(\frac{1}{2}+(v-1)\kappa\right)},
\end{aligned} \tag{A.90}$$

where the second inequality is due to (A.10) and the last inequality follows from the fact that $v > 2$ so that $\sum\gamma(\ell)\ell < \infty$, while $\ell_n \asymp \Delta_n^{-\kappa}$. Then (A.89) and (A.90) imply

$$\left|\frac{(n-k_n+2)\,\alpha_n\sqrt{\Delta_n}}{\theta\psi_0} - \frac{\psi_1}{\theta^2\psi_0}\Gamma_n\right| \leq C\Delta_n^{\frac{1}{2}\wedge(v-1)\kappa} + o(\Delta_n^{1/4}).$$

Since $(v-1)\kappa > 1/4$ we have

$$\Delta_n^{-1/4}\left(\frac{(n-k_n+2)\,\alpha_n\sqrt{\Delta_n}}{\theta\psi_0} - \frac{\psi_1}{\theta^2\psi_0}\Gamma_n\right) \to 0. \tag{A.91}$$

On the other hand, we have by (A.51) that

$$\Delta_n^{-1/4}\left(\frac{\psi_1}{\theta^2\psi_0}\left(\Gamma_n - \sum_{|\ell|\leq\ell_n}\widehat{\gamma(\ell)}_n\right)\right) \xrightarrow{\mathbb{P}} 0. \tag{A.92}$$

Now (A.85) is proven by (A.91) and (A.92).

(4) Proof of (A.86): see Lemma 5.5 of Jacod et al. (2009).

(5) For (A.87), we apply Hölder's inequality and (A.88) to conclude that $\mathbb{E}\Big(\big(\eta'(p)_j^n\big)^2\Big) \leq C_p\Delta_n$. Doob's

inequality and the fact that $K_n^p \le C/(p\sqrt{\Delta_n})$ then together imply

$$\mathbb{E}\Big(\sup_{t\le T}\left(M'(p)_n\right)^2\Big) \le 4\sum_{j=0}^{K_n^p}\mathbb{E}\Big(\left(\eta'(p)_j^n\right)^2\Big) \le C\sqrt{\Delta_n}/p.$$

□

Lemma A.10. *For any $p \ge 2$, the sequence $\Delta_n^{-1/4}M(p)_n$ of processes converges stably in law to*

$$\Upsilon_1(p) = \int_0^1 V(p)_s \mathrm{d}W'_s, \tag{A.93}$$

where W' is as in Theorem 4.3 and $V(p)_t$ is the square root of

$$V(p)_t^2 = \frac{4}{\psi_0^2(p+1)}\left(\Lambda_{00}(p)\theta\sigma_t^4 + 2\Lambda_{01}(p)\frac{\sigma_t^2\Sigma_U}{\theta} + \Lambda_{11}(p)\frac{\Sigma_U^2}{\theta^3}\right). \tag{A.94}$$

Proof. In view of the classical central limit theorems for triangular arrays of martingale differences in, e.g., Theorem IX.7.28 in Jacod and Shiryaev (2003), it suffices to prove the following:

$$\frac{1}{\sqrt{\Delta_n}}\sum_{j=0}^{K_n^p}\left(\mathbb{E}\left(\left(\eta(p)_j^n\right)^2|\mathcal{J}(p)_j^n\right) - \left(\overline{\eta}(p)_j^n\right)^2\right) \xrightarrow{\mathbb{P}} \int_0^1 V(p)_s^2\mathrm{d}s, \tag{A.95}$$

$$\frac{1}{\Delta_n}\sum_{j=0}^{K_n^p}\mathbb{E}\left(\left(\eta(p)_j^n\right)^4|\mathcal{J}(p)_j^n\right) \xrightarrow{\mathbb{P}} 0, \tag{A.96}$$

$$\frac{1}{\Delta_n^{1/4}}\sum_{j=0}^{K_n^p}\mathbb{E}\left(\left(\eta(p)_j^n\right)\Delta(N,p)_j^n\,|\mathcal{J}(p)_j^n\right) \xrightarrow{\mathbb{P}} 0, \tag{A.97}$$

where $\Delta(Z,p)_j^n = Z_{(j+1)(p+1)k_n\Delta_n} - Z_{j(p+1)k_n\Delta_n}$, and N is any bounded martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P})$ orthogonal to W, or $N = W$.

1. Proof of (A.95). Equation (A.66) implies $\mathbb{E}\Big(\left(\overline{\eta}(p)_j^n\right)^2\Big) \le C_p\Delta_n^2$, whence $\frac{1}{\sqrt{\Delta_n}}\sum_{j=0}^{K_n^p}\left(\overline{\eta}(p)_j^n\right)^2 \xrightarrow{\mathbb{P}} 0$. The estimate (A.72), plus Riemann integration, and (18) yield

$$\frac{1}{\sqrt{\Delta_n}}\sum_{j=0}^{K_n^p}\mathbb{E}\left(\left(\eta(p)_j^n\right)^2|\mathcal{J}(p)_j^n\right) \xrightarrow{\mathbb{P}} \int_0^1 V(p)_s^2\mathrm{d}s.$$

2. Proof of (A.96). By (A.65) we have $\Delta_n^{-2}\mathbb{E}\Big(\left(\eta(p)_j^n\right)^4\Big) \le C$ so the Markov inequality gives $\mathbb{E}\left(\left(\eta(p)_j^n\right)^4|\mathcal{J}(p)_j^n\right) = O_p(\Delta_n^2)$. Then (A.96) follows since $K_n^p \le C_p\Delta_n^{-\frac{1}{2}}$.

3. Proof of (A.97). Let $\ell(p)_j^n = j(p+1)k_n$. It is equivalent to prove

$$\Delta_n^{1/4}\sum_{j=0}^{K_n^p}\mathbb{E}\left(\zeta(p)_{\ell(p)_j^n}^n\Delta(N,p)_j^n\,|\mathcal{J}(p)_j^n\right) \xrightarrow{\mathbb{P}} 0.$$

In view of (5.62) of Jacod et al. (2009), it then suffices to prove that

$$L_n(p) := \Delta_n^{1/4} \sum_{j=0}^{K_n^p} \mathbb{E}\left(\sum_{i=\ell(p)_j^n}^{\ell(p)_j^n + pk_n - 1} (\widehat{U}_i^n + 2\overline{X}_i^n \overline{U}_i^n) \Delta(N,p)_j^n \,\middle|\, \mathcal{J}(p)_j^n \right) \xrightarrow{\mathbb{P}} 0, \tag{A.98}$$

where we may assume that N is a square-integrable martingale.

Since N is a process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, the independence of $\mathcal{F}$ and $\mathcal{G}$ implies that

$$\sum_{i=\ell(p)_j^n}^{\ell(p)_j^n + pk_n - 1} \left| \mathbb{E}\left(\widehat{U}_i^n \Delta(N,p)_j^n \,\middle|\, \mathcal{J}(p)_j^n \right) \right| \leq \sqrt{\mathbb{E}\left(\left(\Delta(N,p)_j^n\right)^2 \,\middle|\, \mathcal{F}^n_{\ell(p)_j^n} \right)} \Theta(p)_j^n;$$
$$\sum_{i=\ell(p)_j^n}^{\ell(p)_j^n + pk_n - 1} \left| \mathbb{E}\left(\overline{X}_i^n \overline{U}_i^n \Delta(N,p)_j^n \,\middle|\, \mathcal{J}(p)_j^n \right) \right| \leq \sqrt{\mathbb{E}\left(\left(\Delta(N,p)_j^n\right)^2 \,\middle|\, \mathcal{F}^n_{\ell(p)_j^n} \right)} \overline{\Theta}(p)_j^n;$$

where

$$\Theta(p)_j^n := \sum_{i=\ell(p)_j^n}^{\ell(p)_j^n + pk_n - 1} \left| \mathbb{E}\left(\widehat{U}_i^n \,\middle|\, \mathcal{G}_{\ell(p)_j^n - \lfloor \frac{k_n}{2} \rfloor} \right) \right|; \quad \overline{\Theta}(p)_j^n := \sum_{i=\ell(p)_j^n}^{\ell(p)_j^n + pk_n - 1} \left| \mathbb{E}\left(\overline{U}_i^n \,\middle|\, \mathcal{G}_{\ell(p)_j^n - \lfloor \frac{k_n}{2} \rfloor} \right) \right| \sqrt{\Delta_n^{1/2}}.$$

Note that we have used (A.6) to bound $\mathbb{E}\left(\left(\overline{X}_i^n\right)^2 \,\middle|\, \mathcal{F}^n_{\ell(p)_j^n} \right)$ by $\Delta_n^{1/2}$. We find that

$$L_n(p)^2 \leq \sqrt{\Delta_n} \left(\sum_{j=0}^{K_n^p} \sqrt{\mathbb{E}\left(\left(\Delta(N,p)_j^n\right)^2 \,\middle|\, \mathcal{F}^n_{\ell(p)_j^n} \right)} (\Theta(p)_j^n + 2\overline{\Theta}(p)_j^n) \right)^2.$$

Repeated applications of the JS-Lemma and the independence of $\mathcal{G}$ and $\mathcal{F}$ give

$$\mathbb{E}\Big(\left(\Theta(p)_j^n\right)^2 \Big) \leq \frac{C_p}{k_n^{2(v-1)}}; \quad \mathbb{E}\Big(\left(\overline{\Theta}(p)_j^n\right)^2 \Big) \leq \frac{C_p}{k_n^{2v}}, \tag{A.99}$$

so we have

$$\begin{aligned} \mathbb{E}\Big((L_n(p))^2 \Big) &\leq \sqrt{\Delta_n} \mathbb{E}\left(\sum_{j=0}^{K_n^p} \left(\Delta(N,p)_j^n\right)^2 \right) \mathbb{E}\left(\sum_{j=0}^{K_n^p} \left(\Theta(p)_j^n + 2\overline{\Theta}(p)_j^n\right)^2 \right) \\ &\leq C_p \mathbb{E}\Big((N_1 - N_0)^2 \Big) \Delta_n^{v-1} \to 0. \end{aligned}$$

The first inequality is an application of Cauchy-Schwarz inequality, and the second one is due to the fact that N is a square-integrable martingale, the estimate (A.99) and the fact that $K_n^p \leq C_p/\sqrt{\Delta_n}$.

This completes the proof of Lemma A.10. □

A.4.2 Proof of Theorem 4.3

Proof. We invoke Lemmas A.9 and A.10, which in turn rely on Lemmas A.6, A.7 and A.8. Recalling the decomposition in (A.2), we note that we have proved in Lemma A.9 that

$$\lim_{p\to\infty}\limsup_{n\to\infty}\mathbb{P}(|Q(p)_n|\geq\varepsilon)=0,$$

for any $\varepsilon>0$, where

$$Q(p)_n:=\Delta_n^{-1/4}\left(M'(p)_n+F(p)_n+F'(p)_n+\widehat{C}(p)_n+\widehat{C}'(p)_n+\widehat{C}''_n\right).$$

Lemma A.10 shows convergence of $\Delta_n^{-1/4}M(p)_n$ to $\Upsilon_1(p)$, and for the fixed Brownian motion W' we have that $V(p)_t(\omega)$ converges pointwise to $V_t(\omega)$ so $\Upsilon(p)_1\xrightarrow{\mathbb{P}}\Upsilon_1$. This proves Theorem 4.3. □

A.4.3 Proof of Corollary 4.2

Proof. The result $\widehat{\Sigma}_n\xrightarrow{\mathbb{P}}\int_0^1 V_t^2\mathrm{d}t$ follows from the following convergence in probability results:

$$\widehat{\Sigma}_{U_n}\xrightarrow{\mathbb{P}}\Sigma_U,\quad \widetilde{\mathrm{IV}}_n\xrightarrow{\mathbb{P}}\mathrm{IV};\ \sum_{i=0}^{n-k_n+1}\left(\overline{Y}_i^n\right)^4\xrightarrow{\mathbb{P}}\int_0^1\left(3\theta^2\psi_0^2\sigma_t^4+6\psi_0\psi_1\sigma_t^2\Sigma_U+\frac{3}{\theta^2}\psi_1^2\Sigma_U^2\right)\mathrm{d}t.$$

The first two statements follow from Proposition 3.3 and Theorem 4.3, whereas the last one is due to (5.65) in Jacod et al. (2009) when we replace the asymptotic variance of pre-averaged noise (called α_t in that paper) by Σ_U, and this can be done because of our Lemma A.1. □

A.5 Proofs of the Results in Section 5

A.5.1 Proof of Theorem 5.1

Proof. By Theorem 4.3 we have $\widetilde{\mathrm{IV}}_n\xrightarrow{\mathbb{P}}\mathrm{IV}$, by Proposition 3.3 we have $\widehat{\Sigma}_{U_n}\xrightarrow{\mathbb{P}}\Sigma_U$ and by Proposition 3.1 we have $\widetilde{\Sigma}_{U_n}^{(1)}=\langle Y,Y\rangle(1)_n\xrightarrow{\mathbb{P}}\gamma(0)-\gamma(1)$. Therefore

$$\widetilde{\mathrm{IV}}_n^{(1)}=\widetilde{\mathrm{IV}}_n+\frac{\psi_1(\widehat{\Sigma}_{U_n}-\widetilde{\Sigma}_{U_n}^{(1)})}{\theta^2\psi_0}\xrightarrow{\mathbb{P}}\mathrm{IV}+\frac{\psi_1(\Sigma_U-\gamma(0)+\gamma(1))}{\theta^2\psi_0}.$$

This shows that

$$\widetilde{\gamma(0)}_n^{(2)}-\widehat{\gamma(0)}_n=O_p(j_n\Delta_n),\quad \widetilde{\langle Y,Y\rangle}(j)_n^{(2)}-\widehat{\langle Y,Y\rangle}(j)_n=O_p(\Delta_n j),$$

which gives

$$\begin{aligned}\widetilde{\gamma(j)}_n^{(2)}-\widehat{\gamma(j)}_n&=\left(\widetilde{\gamma(0)}_n^{(2)}-\widetilde{\langle Y,Y\rangle}(j)_n^{(2)}\right)-\left(\widehat{\gamma(0)}_n-\widehat{\langle Y,Y\rangle}(j)_n\right)\ =\ O_p(\Delta_n(j\vee j_n)),\\ \widetilde{\Sigma}_{U_n}^{(2)}-\widehat{\Sigma}_{U_n}&=\widetilde{\gamma(0)}_n^{(2)}-\widehat{\gamma(0)}_n+2\sum_{j=1}^{\ell_n}\left(\widetilde{\gamma(j)}_n^{(2)}-\widehat{\gamma(j)}\right)\ =\ O_p((\ell_n^2\vee j_n\ell_n)\Delta_n).\end{aligned}$$

The asymptotic conditions (14) then imply that $\Delta_n^{-1/4}\left(\widetilde{\Sigma}_{U_n}^{(2)} - \widehat{\Sigma}_{U_n}\right) \xrightarrow{\mathbb{P}} 0$. This proves (43) for $K = 2$ and the consistency $\widetilde{\Sigma}_{\mathrm{IV}_n}^{(2)} \xrightarrow{\mathbb{P}} \int_0^1 V_t^2 \mathrm{d}t$. It also immediately yields $\Delta_n^{-1/4}\left(\widetilde{\mathrm{IV}}_n^{(2)} - \widetilde{\mathrm{IV}}_n\right) \xrightarrow{\mathbb{P}} 0$.

Now assume we have for a certain $k \geq 2$ that

$$\Delta_n^{-1/4}\left(\widetilde{\Sigma}_{U_n}^{(k)} - \widehat{\Sigma}_{U_n}\right) \xrightarrow{\mathbb{P}} 0; \tag{A.100}$$

$$\Delta_n^{-1/4}\left(\widetilde{\mathrm{IV}}_n^{(k)} - \widetilde{\mathrm{IV}}_n\right) \xrightarrow{\mathbb{P}} 0; \tag{A.101}$$

$$\widetilde{\Sigma}_{\mathrm{IV}_n}^{(k)} \xrightarrow{\mathbb{P}} \int_0^1 V_t^2 \mathrm{d}t. \tag{A.102}$$

A direct calculation shows

$$\widetilde{\Sigma}_{U_n}^{(k+1)} - \widetilde{\Sigma}_{U_n}^{(k)} = \left(-\frac{(2\ell_n+1)j_n}{2(n-j_n+1)} + \sum_{j=1}^{\ell_n} \frac{j}{n-j+1}\right)(\widetilde{\mathrm{IV}}_n^{(k)} - \widetilde{\mathrm{IV}}_n^{(k-1)}) = O_p((\ell_n^2 \vee j_n\ell_n)\Delta_n);$$

$$\widetilde{\mathrm{IV}}_n^{(k+1)} - \widetilde{\mathrm{IV}}_n^{(k)} = \frac{\psi_1\left(\widetilde{\Sigma}_{U_n}^{(k)} - \widetilde{\Sigma}_{U_n}^{(k+1)}\right)}{\theta^2\psi_0} = O_p((\ell_n^2 \vee j_n\ell_n)\Delta_n).$$

Assumption (14) then implies that (A.100) and (A.101) hold for $k+1$ as well, and (A.102) then follows. This proves Theorem 5.1. □

B Additional Simulation Studies

In this section, we provide additional Monte Carlo simulation results that assess the effects of price discreteness and correlation between X and U. Price discreteness renders dependence between X and U. The results in Section B.1 show that the presence of minimal ticks has relatively little impact on the estimation of the moments of noise and the IV. Furthermore, the results in Section B.2 show that in the situation when X and U are correlated our multi-step estimators still appear to be performing well.

B.1 Price Discreteness

We consider a setting in which the observed price is rounded to 1 cent. The observed logarithmic price is then given by:

$$Y_t^{\mathrm{rd}} = \log\left([100\exp(Y_t)]/100\right), \tag{B.1}$$

where $[x]$ denotes the integer that is closest to x. Now the microstructure noise has two components:

$$U_i^{\mathrm{rd}} = Y_{i\Delta_n}^{\mathrm{rd}} - X_{i\Delta_n} = \underbrace{Y_{i\Delta_n}^{\mathrm{rd}} - Y_{i\Delta_n}}_{\text{error due to discreteness}} + \underbrace{Y_{i\Delta_n} - X_{i\Delta_n}}_{\text{error due to market microstructure}}. \tag{B.2}$$

Figure B.1 compares our two-step estimators of the second moments of U^{rd} to the true values for the model setup of Section 6. The two-step estimators still yield accurate estimates, although there is a small bias. In the estimation of the integrated volatility, we have a bias of 4.47×10^{-5} and a standard deviation of 3.55×10^{-5}; these are relatively small compared to the expected value of the integrated volatility which is 4.44×10^{-4}.

B.2 Correlation between X and U

We also provide simulation evidence on the robustness of our estimators when dependence between X and U is introduced by choosing a fixed correlation $\rho_{\epsilon W}$ between the process ϵ in (45) and the increments of the Brownian motion W. Table B.1 shows the centered means and standard deviations of $\widetilde{\mathrm{IV}}_n^{(2)}$. Results are shown for the cases $\rho_{\epsilon W} = 0$, $\rho_{\epsilon W} = 0.7$ and $\rho_{\epsilon W} = -0.7$, and for three different values of the tuning parameter: $\theta = 0.4$, $\theta = 0.6$, and the value $\theta = \theta^*$ defined in (28). The results show that our estimator is relatively insensitive to the choice of the tuning parameter θ and to the correlation between X and U for this model specification.

In a second simulation experiment, we investigate the performance of our two-step estimators for the second moments of noise when the increments of the Brownian motion W and the noise component e in (45) are correlated. The fixed correlation coefficient ρ_{eW} was taken to be either 1 or -1. The results in Figure B.2 show that the biases in the estimates are very small, both for a fixed value of θ and for the optimized value θ^*.

ι	-0.7		-0.3		0		0.3		0.7	
$\theta = 0.4$										
$\rho_{\epsilon W} = 0$	-1.33	(3.72)	-0.96	(3.71)	-0.62	(3.71)	-0.14	(3.72)	1.00	(3.78)
$\rho_{\epsilon W} = 0.7$	-1.39	(3.71)	-1.01	(3.72)	-0.60	(3.72)	0.04	(3.74)	2.00	(3.88)
$\rho_{\epsilon W} = -0.7$	-1.26	(3.71)	-0.90	(3.70)	-0.62	(3.70)	-0.31	(3.71)	0.02	(3.73)
$\theta = 0.6$										
$\rho_{\epsilon W} = 0$	-1.00	(4.33)	-0.93	(4.33)	-0.86	(4.33)	-0.77	(4.34)	-0.56	(4.38)
$\rho_{\epsilon W} = 0.7$	-1.00	(4.33)	-0.93	(4.33)	-0.85	(4.34)	-0.72	(4.35)	-0.35	(4.43)
$\rho_{\epsilon W} = -0.7$	-0.99	(4.33)	-0.92	(4.33)	-0.87	(4.33)	-0.81	(4.33)	-0.76	(4.34)
$\theta = \theta^*$										
$\rho_{\epsilon W} = 0$	-1.18	(3.87)	-0.97	(3.91)	-0.80	(3.91)	-0.59	(3.96)	-0.24	(4.13)
$\rho_{\epsilon W} = 0.7$	-1.21	(3.88)	-0.99	(3.90)	-0.79	(3.92)	-0.48	(4.00)	0.00	(4.28)
$\rho_{\epsilon W} = -0.7$	-1.15	(3.90)	-0.96	(3.89)	-0.82	(3.90)	-0.67	(3.92)	-0.59	(3.95)

Table B.1: Estimation of the IV using $\widetilde{\mathrm{IV}}_n^{(2)}$ when X and U are correlated. The numbers represent the centered means with standard deviations between parentheses, based on 1,000 simulations for each scenario. All numbers in the table are multiplied by 10^5. The time step is $\Delta_n = 1$ sec and the number of observations n is 23,400. For the tuning parameters we took $j_n = 20$ and $\ell_n = 10$ while the value of θ varies, as shown in the first column of the table.

C Empirical Study of Transaction Data for General Electric

We collect 2,721,475 transaction prices of General Electric (GE) over the month January 2011. On average there are 5.8 observations per second. In contrast to the analysis of Citigroup transaction prices in Sections 7.2 and 7.3, bias correction plays a very pronounced role here. Despite the high data frequency, the finite sample bias can be very significant if the underlying noise-to-signal ratio is small (recall Remark 3.3). This is indeed the case as Figure C.1 reveals: compared with Citigroup, the data frequency of the General Electric sample is typically lower but the noise-to-signal ratio is also (much) smaller. While the data frequency is immediately available, the noise-to-signal ratio is latent. Therefore, one should always be wary to rely solely on asymptotic theory in practice.

The top panel of Figure C.2 shows that both the realized volatility (RV) and local averaging (LA) estimators indicate that the noise is strongly autocorrelated, while the bias corrected realized volatility (BCRV) estimator reveals that the noise is only weakly dependent. Such a pattern also appears in our simulation study, where we have seen that it is the finite sample bias that induces this discrepancy. Since the dependence in noise is quite weak, we would expect the estimators $\widetilde{\mathrm{IV}}_n^{(1)}$ and $\widetilde{\mathrm{IV}}_n^{(2)}$ to be close to each other, if the latter is accurate. This is indeed the case, as the bottom panel of Figure C.2 shows. However, the other two estimators $\widetilde{\mathrm{IV}}_n$ and $\widetilde{\mathrm{IV}}_n^{\mathrm{JLZ}}$, which don't apply finite sample bias corrections, seem to be biased downwards.

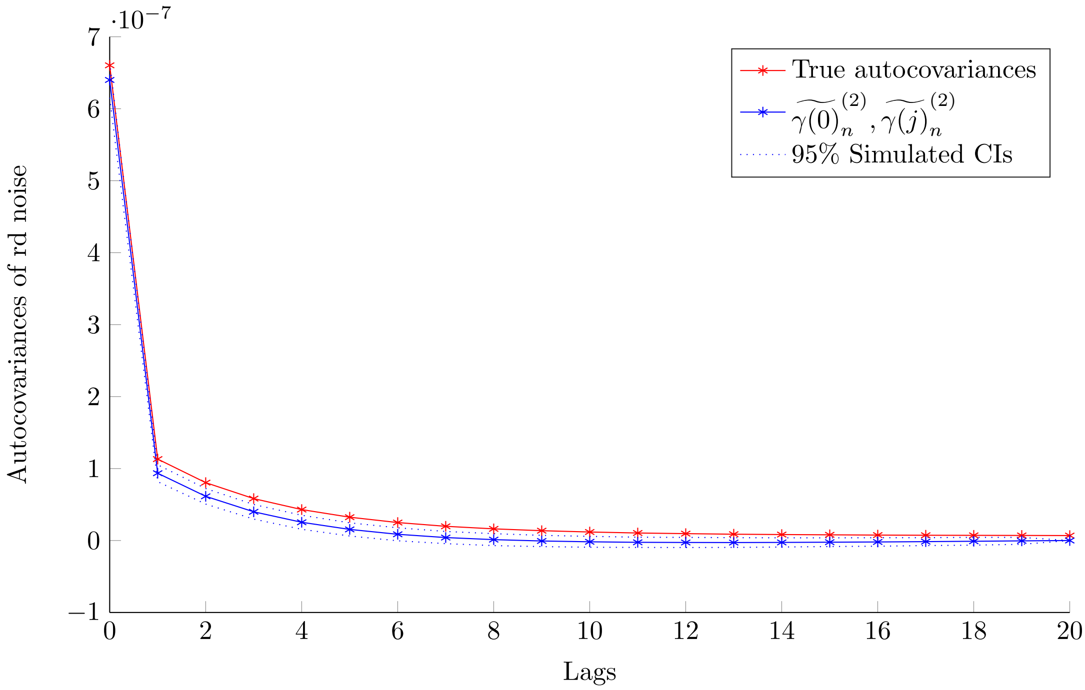

Figure B.1: Estimation of the autocovariances of microstructure noise with rounded prices as specified in (B.1) and (B.2) for the model setup of Section 6. The estimators $\widetilde{\gamma(0)}_n^{(2)}, \widetilde{\gamma(j)}_n^{(2)}$ are defined in (38) and (39). The AR(1)-coefficient of U equals $\iota = 0.7$. The number of simulations is 1,000 and the time step is $\Delta_n = 0.2$ sec. The tuning parameters are $j_n = 20$ and $\ell_n = 10$ and θ is selected according to (28). The "true autocovariances" were determined as the means of the 1,000 sample autocovariances of U^{rd}.

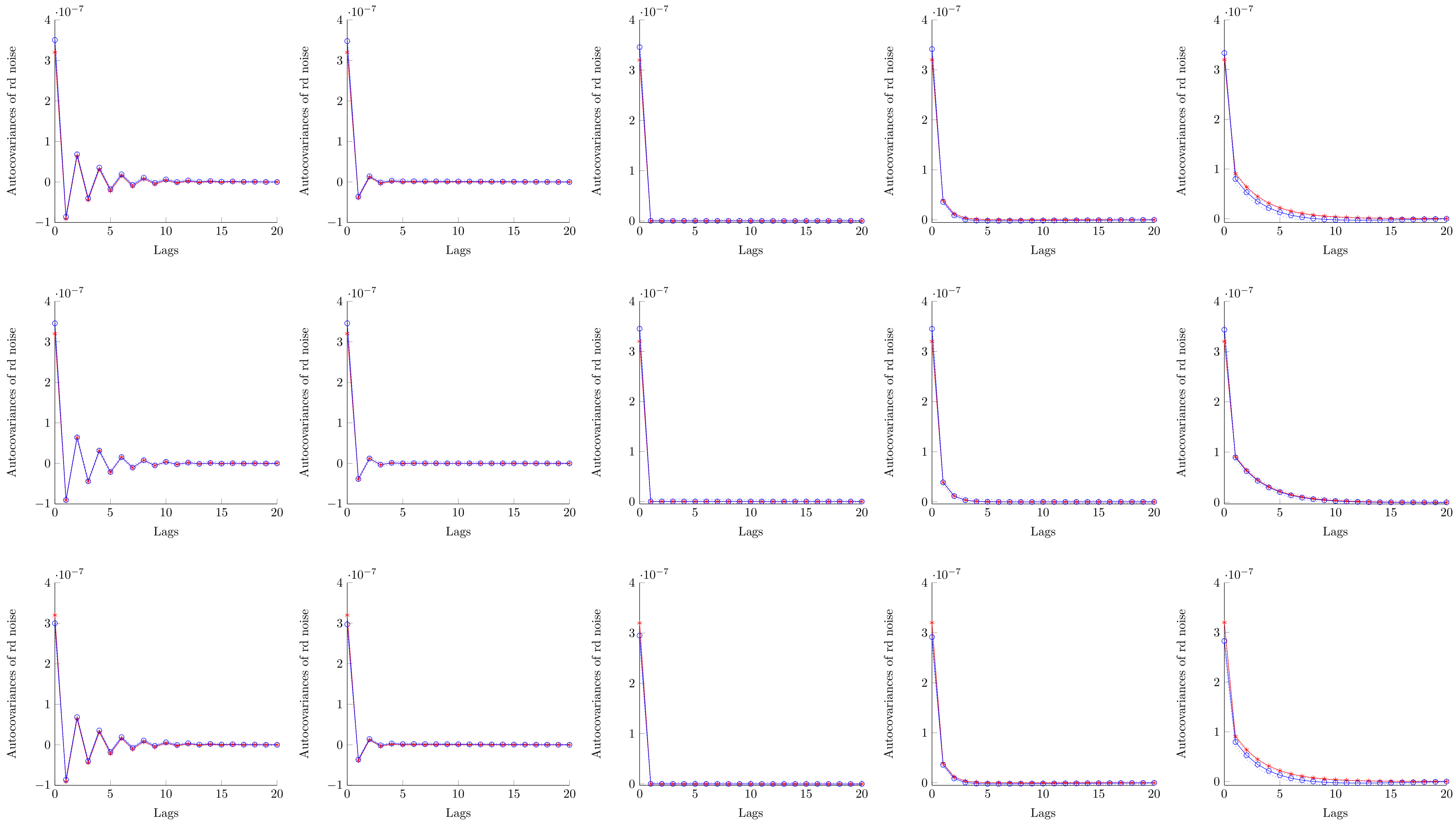

Figure B.2: Estimation of the autocovariances of noise using $\widetilde{\gamma(0)}_n^{(2)}, \widetilde{\gamma(j)}_n^{(2)}$ when X and U are correlated. The red stars are the true values of the autocovariances of noise. The blue circles are the mean estimates of our two-step estimators and the dashed lines are the 95% simulated confidence intervals. The number of simulations is 1,000. $\Delta = 0.2$ sec and the number of observations is 468,000. The tuning parameter of the RV estimator is $j_n = 20$ and $\ell_n = 10$. The three panels, from top to bottom correspond to the specifications $\rho_{eW} = 1$, $\theta = \theta^*$, $\rho_{eW} = 1$, $\theta = 0.6$ and $\rho_{eW} = -1$, $\theta = \theta^*$. The correlation parameter ι varies from left to right in the panels; we show results for ι equal to -0.7, -0.3, 0, 0.3 and 0.7.

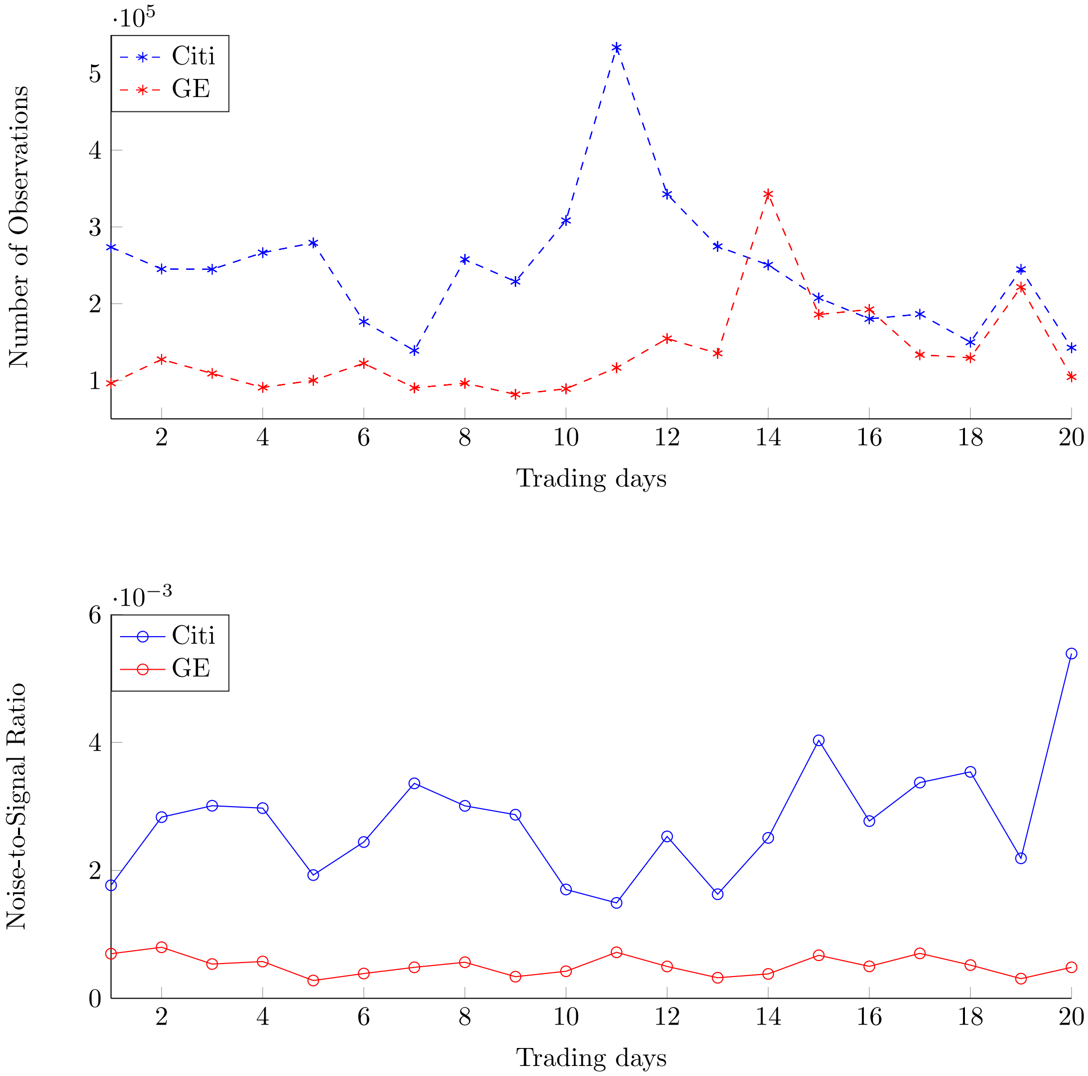

Figure C.1: Number of daily observations of transaction prices (top panel) and noise-to-signal ratio (bottom panel) for Citigroup (C) and General Electric (GE). Sample period: January, 2011, consisting of 20 trading days. In the bottom panel, the noise-to-signal ratio, $\frac{\Sigma_U^2}{\int_0^1 \sigma_s^2 \mathrm{d}s}$, is estimated by $\frac{\widetilde{\Sigma}_{U_n}^{(2)}}{\widetilde{\mathrm{IV}}_n^{(2)}}$, where $\widetilde{\Sigma}_{U_n}^{(2)}$ and $\widetilde{\mathrm{IV}}_n^{(2)}$ are defined in (40) and (41), respectively. We set $j_n = 30$, $\ell_n = 10$ and θ is selected according to (28).

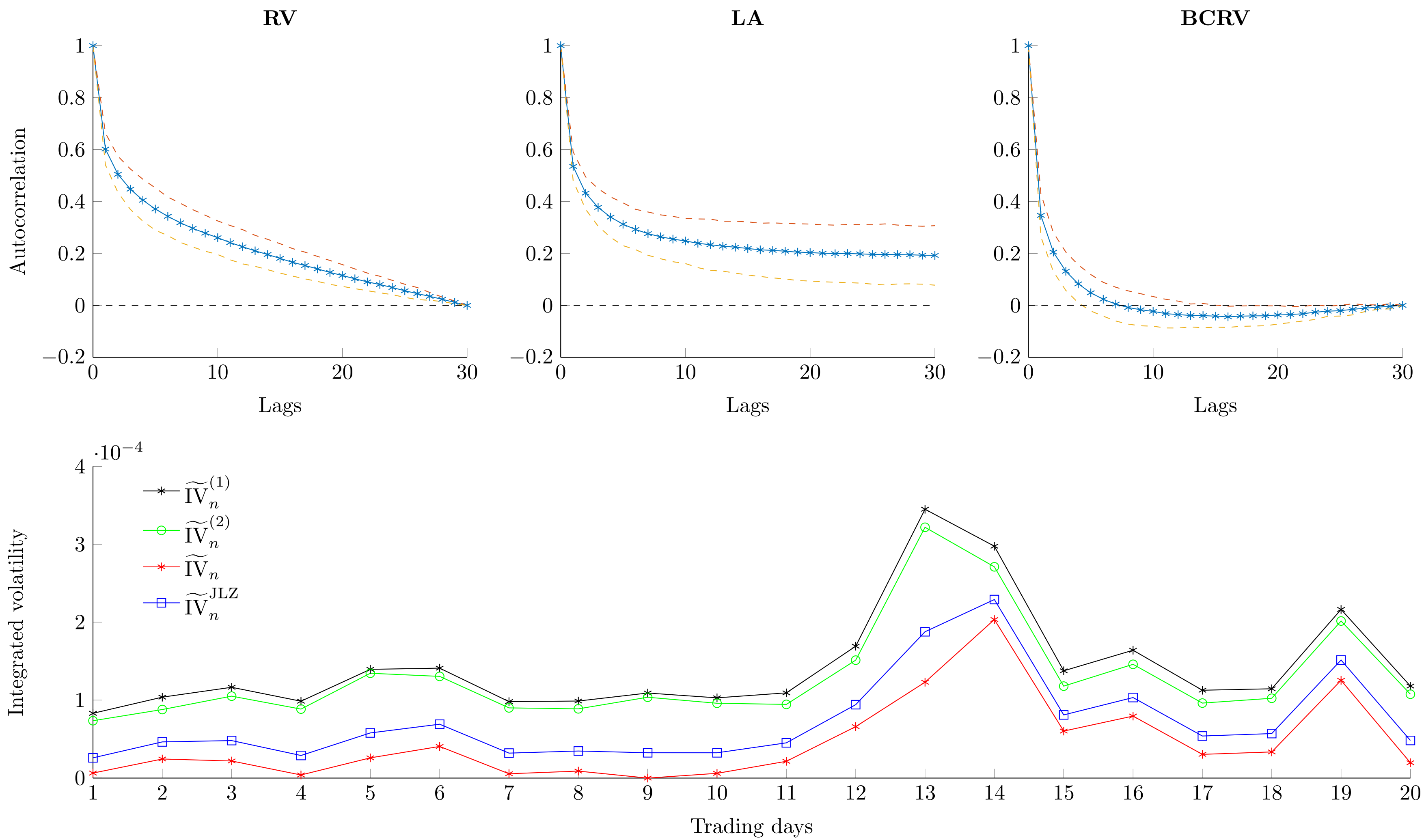

Figure C.2: Autocorrelations of noise and integrated volatility based on transaction data for General Electric (GE). Sample period: January, 2011, consisting of 20 trading days. On average there are 5.8 observations per second in the sample. Top panel: From the left to the right, we display the realized volatility (RV), local averaging (LA), and the bias corrected realized volatility (BCRV) estimators of the autocorrelations of noise against the number of lags j. The three estimators are applied to and then averaged over each of the 20 trading days. The stars indicate the means of the 20 estimates. The dashed lines are 2 standard deviations away from the mean. Bottom panel: Estimation of the integrated volatility. The estimators $\widetilde{\mathrm{IV}}_n^{(1)}$, $\widetilde{\mathrm{IV}}_n^{(2)}$, and $\widetilde{\mathrm{IV}}_n$ are given by (36), (41), and (25). The $\widetilde{\mathrm{IV}}_n^{\mathrm{JLZ}}$ estimator is proposed in Jacod et al. (2019). We set $j_n = 30$, $\ell_n = 10$ and θ is selected according to (28).

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